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Hyperpolyadic Structures

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Abstract: We introduce a new class of division algebras, the hyperpolyadic algebras, which correspond to the binary division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ without considering new elements. First, we use the matrix polyadization procedure proposed earlier which increases the dimension of the algebra. The algebras obtained in this way obey binary addition and a nonderived n-ary multiplication and their subalgebras are division n-ary algebras. For each invertible element, we define a new norm which is polyadically multiplicative, and the corresponding map is a n-ary homomorphism. We define a polyadic analog of the Cayley–Dickson construction which corresponds to the consequent embedding of monomial matrices from the polyadization procedure. We then obtain another series of n-ary algebras corresponding to the binary division algebras which have a higher dimension, which is proportional to the intermediate arities, and which are not isomorphic to those obtained by the previous constructions. Second, a new polyadic product of vectors in any vector space is defined, which is consistent with the polyadization procedure using vectorization. Endowed with this introduced product, the vector space becomes a polyadic algebra which is a division algebra under some invertibility conditions, and its structure constants are computed. Third, we propose a new iterative process (we call it the "imaginary tower"), which leads to nonunital nonderived ternary division algebras of half the dimension, which we call "half-quaternions" and "half-octonions". The latter are not the subalgebras of the binary division algebras, but subsets only, since they have different arity. Nevertheless, they are actually ternary division algebras, because they allow division, and their nonzero elements are invertible. From the multiplicativity of the introduced "half-quaternion" norm, we obtain the ternary analog of the sum of two squares identity. We show that the ternary division algebra of imaginary "half-octonions" is unitless and totally associative.

Keywords: hypercomplex algebra; ternary algebra; n-ary algebra; querelement; quaternion; octonion; Cayley–Dickson construction; division algebra; structure constant; vector space; vectorization; vector multiplication

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1. Introduction

The field extension is a fundamental concept of algebra [1–3] and number theory [4–6]. Informally, the main idea is to enlarge a given structure in a special way (using elements from outside the underlying set) and try to obtain a resulting algebraic structure with "good" properties. One of the first well-known examples is the field of complex numbers $\mathbb{C}$, which is a simple field extension of real numbers $\mathbb{R}$. The direct generalization of this construction leads to the hypercomplex numbers (see, e.g., [7,8]) defined as finite D-dimensional algebras $\mathbb{A}$ over the reals with a special basis (with squares restricted to $0, \pm 1$). Among numerous versions of hypercomplex number systems [9,10] (for a modern review, see, e.g., [11,12]), only the complex numbers $\mathbb{A} = \mathbb{C}$ $(D = 2)$, quaternions $\mathbb{A} = \mathbb{H}$ $(D = 4)$, and octonions $\mathbb{A} = \mathbb{O}$ $(D = 8)$ are classical division algebras (with no zero divisors or nilpotents) [13–15], and the latter two can be obtained via the Cayley–Dickson doubling procedure [16–18].

In this paper, we construct nonderived hyperpolyadic structures corresponding to the above division algebras without introducing new elements (as in, e.g., [19,20]). Recall the numerous applications of higher arity structures in physics [21–23], in particular in particle dynamical models [24–26] and supersymmetry [27–29]. There are plenty of n-ary generalizations of associative and Lie algebras, for examples and reviews, see [30–33].

Here, we will first use the matrix polyadization procedure proposed by the author in [34]. We show that the polyadic analog of the Cayley–Dickson construction can only lead to non-division algebras of higher dimensions than the initial division algebras. For the n-ary algebras thus obtained, we introduce a new norm which is polyadically multiplicative and is well defined for invertible elements.

Second, we propose a new polyadic product of vectors in any vector space, which is consistent with the polyadization procedure using vectorization. Endowed with this product, the vector space becomes a polyadic algebra, which is a division algebra under some invertibility conditions. Its structure constants are computed, and a numerical example is given.

Third, on the subsets of the binary division algebras, we propose another new construction (we call it the "imaginary tower") and an iterative process which naturally gives the corresponding nonderived ternary division algebras of half a dimension. The latter are not subalgebras, because they have a different multiplication and different arity than the initial algebras. We call the nonunital ternary algebras obtained in this way "half-quaternions" and "half-octonions". They are actually division algebras, because nonzero elements are invertible and thus allow division. From the multiplicativity of the "half-quaternion" norm, we obtain the ternary analog of the sum of two squares identity. Finally, we show that the unitless ternary division algebra of imaginary "half-octonions" is ternary totally associative.

2. Preliminaries

Here, we briefly remind the reader of notation from [34]. A (one-set) polyadic algebraic structure $\mathscr{A}$ is a set A closed with respect to polyadic operations (or n-ary multiplication) $\mu^{[n]} : \mathsf{A}^n \to \mathsf{A}$ (n-ary magma). We denote polyads [35] as $\vec{a} = \vec{a}^{(k)} = (a_1, \dots, a_k)$, $a_i \in \mathsf{A}$, and $a^k = \Big(\overbrace{a, \dots, a}^{k}\Big)$, $a \in \mathsf{A}$ (usually, the value of k follows from the context). A (positive) polyadic power is

$$a^{\langle \ell_\mu \rangle} = \left(\mu^{[n]}\right)^{\circ \ell_\mu} \left[a^{\ell_\mu (n-1)+1}\right], \quad a \in \mathsf{A},\ \ell_\mu \in \mathbb{N}. \tag{1}$$

A polyadic ℓ_μ-idempotent (or idempotent for $\ell_\mu = 1$) is defined by $a^{\langle \ell_\mu \rangle} = a$. A polyadic zero is defined by $\mu^{[n]}[\vec{a}, z] = z$, $z \in \mathsf{A}$, $\vec{a} \in \mathsf{A}^{n-1}$, where z can be on any place. An element of a polyadic algebraic structure a is called ℓ_μ-nilpotent (or nilpotent for $\ell_\mu = 1$), if there exist ℓ_μ such that $a^{\langle \ell_\mu \rangle} = z$. A polyadic (or n-ary) identity (or neutral element) is defined by

$$\mu^{[n]}\left[a, e^{n-1}\right] = a, \quad \forall a \in \mathsf{A}, \tag{2}$$

where a can be on any place on the l.h.s. of (2). In addition, there exist neutral polyads (usually not unique) satisfying

$$\mu^{[n]}[a, \vec{n}] = a, \quad \forall a \in \mathsf{A}. \tag{3}$$

A one-set polyadic algebraic structure $\left\langle \mathsf{A} \mid \mu^{[n]} \right\rangle$ is totally associative, if

$$\left(\mu^{[n]}\right)^{\circ 2}\left[\vec{a}, \vec{b}, \vec{c}\right] = \mu^{[n]}\left[\vec{a}, \mu^{[n]}\left[\vec{b}\right], \vec{c}\right] = invariant, \tag{4}$$

with respect to the placement of the internal multiplication on any of the n places, and $\vec{a}, \vec{b}, \vec{c}$ are polyads of the necessary sizes.

A polyadic semigroup $\mathcal{S}^{[n]}$ is a one-set and one-operation structure in which $\mu^{[n]}$ is totally associative. A polyadic structure is (totally) commutative, if $\mu^{[n]} = \mu^{[n]} \circ \sigma$, for all $\sigma \in S_n$. A polyadic structure is solvable, if for all polyads b, c and an element x, one can (uniquely) resolve the Equation (with respect to x) for $\mu^{[n]}\left[\vec{b}, x, \vec{c}\right] = a$, where x can be on any place, and $\vec{b}, \vec{c}$ are polyads of the needed lengths. A solvable polyadic structure is called n-ary quasigroup [36]. An associative polyadic quasigroup is called a n-ary (or polyadic) group $\mathcal{G}^{[n]}$ (for a review, see, e.g., [37]). In an n-ary group, the only solution of

$$\mu^{[n]}\left[a^{n-1}, \tilde{a}\right] = a, \quad a, \tilde{a} \in \mathsf{A}, \tag{5}$$

is called the querelement (the polyadic analog of an inverse) of a and denoted by $\tilde{a}$ [38], where $\tilde{a}$ can be on any place. The relation (5) can be considered as a definition of the unary queroperation $\bar{\mu}^{(1)}[a] = \tilde{a}$ [39].

For further details and references, see [34].

3. Matrix Polyadization

Let us briefly (just to establish notation and terminology) recall that the $2-,4-,8$-dimensional algebras are the only hypercomplex extensions of the reals $\mathbb{A} = \mathbb{R}$ (Hurwitz's theorem for composition algebras) $\mathbb{A} = \mathbb{D} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ which are normed division algebras. The first two are associative (and can be represented by matrices), and only $\mathbb{C}$ is commutative (being a field). We use the unified notation $\mathbf{z} \in \mathbb{D}$, and if we need to distinguish and concretize, the standard parametrization will be exploited $\mathbb{C} \ni \mathbf{z} = \mathbf{z}_{(2)} = a + b\mathbf{i}$ and $\mathbb{H} \ni \mathbf{z} = \mathbf{z}_{(4)} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, etc., $a, d, c, d \in \mathbb{R}$. The standard Euclidean norm (2-norm) $\|\mathbf{z}\| = \sqrt{\mathbf{z}^* \mathbf{z}}$ (where the conjugate is $\mathbf{z}^*_{(2)} = a - b\mathbf{i}$, etc.), for $\mathbf{z} \in \mathbb{D}$ (which for $\mathbb{C}$ coincides with the modulus $\left\|\mathbf{z}_{(2)}\right\| = \left|\mathbf{z}_{(2)}\right| = \sqrt{a^2 + b^2}$, and $\left\|\mathbf{z}_{(4)}\right\| = \sqrt{a^2 + b^2 + c^2 + d^2}$, etc.) has the properties

$$\|\mathbf{1}\| = 1, \tag{6}$$

$$\|\lambda \mathbf{z}\| = |\lambda| \|\mathbf{z}\|, \tag{7}$$

$$\|\mathbf{z}' + \mathbf{z}''\| \leq \|\mathbf{z}'\| + \|\mathbf{z}''\|, \quad 1, \lambda \in \mathbb{R}, \ \mathbf{1}, \mathbf{z}, \mathbf{z}', \mathbf{z}'' \in \mathbb{D}, \tag{8}$$

and is multiplicative

$$\|\mathbf{z}_1 \mathbf{z}_2\| = \|\mathbf{z}_1\| \|\mathbf{z}_2\| \in \mathbb{R}_{\geqslant 0}, \quad \mathbf{z} \in \mathbb{D}, \tag{9}$$

such that the corresponding mapping $\mathbb{D} \to \mathbb{R}_{\geqslant 0}$ is a homomorphism. Each nonzero element of the above normed unital algebra has the multiplicative inverse $\mathbf{z}^{-1}\mathbf{z} = \mathbf{1}$, because the norm vanishes only for $\mathbf{z} = 0$ and there are no zero divisors, and therefore, from $\|\mathbf{z}\|^2 = \mathbf{z}^* \mathbf{z} \in \mathbb{R}_{\geqslant 0}$, it follows that

$$\mathbf{z}^{-1} = \frac{\mathbf{z}^*}{\|\mathbf{z}\|^2}, \quad \mathbf{z} \in \mathbb{D} \setminus \{0\}. \tag{10}$$

To construct the polyadic analogs of the binary hypercomplex algebras $\mathbb{A}$ and, in particular, of the binary division algebras $\mathbb{D}$ (over $\mathbb{R}$), we use the polyadization procedure proposed in [34] (called there block-matrix polyadization). It is based on the general structure theorem for polyadic rings (a generalization of the Wedderburn theorem): any simple $(2, n)$-ring is isomorphic to the ring of special cyclic shift block-matrices (of the shape (11)) over a division ring [40].

Let us introduce the $(n-1) \times (n-1)$ cyclic shift weighted matrix with the elements from the algebra $\mathbb{A}$

$$\mathbf{Z} = \mathbf{Z}^{[n]} = \begin{pmatrix} 0 & z_1 & \dots & 0 & 0 \\ 0 & 0 & z_2 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & z_{n-2} \\ z_{n-1} & 0 & \dots & 0 & 0 \end{pmatrix}, \quad z_i \in \mathbb{A}. \tag{11}$$

The matrices of such a shape, i.e., (11) play a considerable role in coding [41] and were intensively studied in [42,43]. Here, we will apply the cyclic shift weighted matrices to the polyadization procedure introduced in [34].

The set of matrices of the form (11) is closed with respect to the ordinary product $(\cdot)$ of exactly n matrices, but not of fewer. Therefore, we can define the n-ary multiplication [34]

$$\mu_{\mathbf{Z}}^{[n]}\left[\overbrace{\mathbf{Z}',\mathbf{Z}''\dots\mathbf{Z}'''}^{n}\right] = \mathbf{Z}'\cdot\mathbf{Z}''\cdot\ldots\cdot\mathbf{Z}''' = \mathbf{Z} = \mathbf{Z}^{[n]}, \tag{12}$$

which is nonderived in the sense that the binary (and all $\leq n-1$) products of $\mathbf{Z}^{[n]}$'s are outside the set (11).

Remark 1. *The binary addition in the algebra $\mathbb{A}$ transfers in the standard way to matrix addition (as component-wise addition), and so we will mostly pay attention to the multiplicative part, implying that the addition of $\mathbf{Z}$-matrices (11) is always binary.*

Definition 1. *We call the following new algebraic structure*

$$\mathbb{A}^{[n]} = \left\langle \left\{ \mathbf{Z}^{[n]} \right\} \mid (+), \mu_{\mathbf{Z}}^{[n]} \right\rangle \tag{13}$$

a n-ary hypercomplex algebra $\mathbb{A}^{[n]}$ which corresponds to the binary hypercomplex algebra $\mathbb{A}$ by the matrix polyadization procedure.

Proposition 1. *The dimension $D^{[n]}$ of the n-ary hypercomplex algebra $\mathbb{A}^{[n]}$ is*

$$D^{[n]} = \dim \mathbb{A}^{[n]} = D(n-1). \tag{14}$$

Proof. It obviously follows from (11). □

Proposition 2. *If the binary hypercomplex algebra $\mathbb{A}$ is associative (for the dimensions $D=1$, $D=2$ and $D=4$), the n-ary multiplication (12) in components has the cyclic product form* [34]

$$\begin{aligned} \overbrace{z_1' z_2'' \dots z_{n-1}''' z_1''''}^{n} &= z_1, \\ \overbrace{z_2' z_3'' \dots z_1''' z_2''''}^{n} &= z_2, \\ &\vdots \\ \overbrace{z_{n-1}' z_1'' \dots z_{n-2}''' z_{n-1}''''}^{n} &= z_{n-1}, \qquad z_i, z_i', \dots, z_i''', z_i'''' \in \mathbb{A}. \end{aligned} \tag{15}$$

Proof. This follows from (11) and (12). □

Remark 2. *The cycled product (15) can be treated as a n-ary extension of the Jordan pair* [44,45]*, which is different from* [46].

Proposition 3. *If $\mathbb{A}$ is unital, then $\mathbb{A}^{[n]}$ contains a n-ary unit (polyadic identity (2)), being the permutation (cyclic shift) matrix of the form*

$$\boldsymbol{E}^{[n]} = \begin{pmatrix} 0 & \mathbf{1} & \dots & 0 & 0 \\ 0 & 0 & \mathbf{1} & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{A}^{[n]}, \quad \mathbf{1} \in \mathbb{A}. \tag{16}$$

Proof. It follows from (11), (12), and (15). □

Consider now the polyadization of the hypercomplex two-dimensional algebra of dual numbers.

Example 1 (4-ary dual numbers)**.** *The commutative and associative two-dimensional algebra $\mathbb{A}_{du} = \langle \{z\} \mid (+), (\cdot) \rangle$ is defined by the element $\boldsymbol{z} = a + b\boldsymbol{\varepsilon}$ with $\boldsymbol{\varepsilon}^2 = 0$, $a, b \in \mathbb{R}$. The binary multiplication $\mu_{\mathsf{v}}^{[2]}$ of pairs (2-tuples) $\mathsf{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{A}_{du}^{tu} = \mathbb{A}_{du}^{[2],tu} = \left\langle \{\mathsf{v}_2\} \mid (+), \mu^{[2]} \right\rangle$ (addition is component-wise, see Remark 1) is*

$$\mu_{\mathsf{v}}^{[2]}\left[\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} a'' \\ b'' \end{pmatrix}\right] = \begin{pmatrix} a'a'' \\ a'b'' + b'a'' \end{pmatrix} \in \mathbb{A}_{du}^{tu}, \quad a', a'', b', b'' \in \mathbb{R}. \tag{17}$$

It follows from (17) that $\mathbb{A}_{du}^{tu}$ (and so also $\mathbb{A}_{du}$) constitutes non-division algebra, because it contains idempotents and zero divisors (e.g., $\begin{pmatrix} 0 \\ b \end{pmatrix}^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$).

Using the matrix polyadization procedure, we construct a 4-ary algebra $\mathbb{A}_{du}^{[4]} = \left\langle \left\{ \boldsymbol{Z}^{[4]} \right\} \mid (+), \boldsymbol{\mu}^{[4]} \right\rangle$ of dimension $D^{[4]} = 6$ (see (14)) by introducing the following 3×3 cyclic shift weighted matrix (11)

$$\boldsymbol{Z}^{[4]} = \begin{pmatrix} 0 & z_1 & 0 \\ 0 & 0 & z_2 \\ z_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 + b_1\boldsymbol{\varepsilon} & 0 \\ 0 & 0 & a_2 + b_2\boldsymbol{\varepsilon} \\ a_3 + b_3\boldsymbol{\varepsilon} & 0 & 0 \end{pmatrix} \in \mathbb{A}_{du}^{[4]}, \quad z_1, z_2, z_3 \in \mathbb{A}_{du}. \tag{18}$$

The cyclic product of the components (15) becomes

$$\begin{aligned} &z_1' z_2'' z_3''' z_1'''' = z_1, \\ &z_2' z_3'' z_1''' z_2'''' = z_2, \\ &z_3' z_1'' z_2''' z_3'''' = z_3, \qquad z_i, z_i', z_i'', z_i''', z_i'''' \in \mathbb{A}_{dual}. \end{aligned} \tag{19}$$

In terms of 6-tuples v_6 over $\mathbb{R}$ (cf. (17)), the 4-ary multiplication $\mu^{[4]}$ in $\mathbb{A}_{du}^{[4],tu} = \left\langle \{\mathsf{v}_6\} \mid (+), \mu_{\mathsf{v}}^{[4]} \right\rangle$ has the form

$$\mu_{\mathsf{v}}^{[4]}\left[\begin{pmatrix} a_1' \\ b_1' \\ a_2' \\ b_2' \\ a_3' \\ b_3' \end{pmatrix}, \begin{pmatrix} a_1'' \\ b_1'' \\ a_2'' \\ b_2'' \\ a_3'' \\ b_3'' \end{pmatrix}, \begin{pmatrix} a_1''' \\ b_1''' \\ a_2''' \\ b_2''' \\ a_3''' \\ b_3''' \end{pmatrix}, \begin{pmatrix} a_1'''' \\ b_1'''' \\ a_2'''' \\ b_2'''' \\ a_3'''' \\ b_3'''' \end{pmatrix}\right]$$
$$= \begin{pmatrix} a_1' a_2'' a_3''' a_1'''' \\ a_1' a_2'' a_3''' b_1'''' + a_1' a_2'' b_3''' a_1'''' + a_1' b_2'' a_3''' a_1'''' + b_1' a_2'' a_3''' a_1'''' \\ a_2' a_3'' a_1''' a_2'''' \\ a_2' a_1''' a_3'' b_2'''' + a_2' a_3'' b_1''' a_2'''' + a_2' b_3'' a_1''' a_2'''' + b_2' a_3'' a_1''' a_2'''' \\ a_3' a_1'' a_2''' a_3'''' \\ a_3' a_1'' a_2''' b_3'''' + a_3' a_1'' b_2''' a_3'''' + a_3' b_1'' a_2''' a_3'''' + b_3' a_1'' a_2''' a_3'''' \end{pmatrix}, \tag{20}$$

which is nonderived and noncommutative due to braidings in the cyclic product (19). The polyadic unit (4-ary unit) in the 4-ary algebra of 6-tuples $\mathsf{e}_6^{[4]}$ is defined by the equation (see (2))

$$\mu_{\mathsf{v}}^{[4]}\left[\mathsf{v}_6, \mathsf{e}_6^{[4]}, \mathsf{e}_6^{[4]}, \mathsf{e}_6^{[4]}\right] = \mathsf{v}_6, \quad \forall \mathsf{v}_6 \in \mathbb{A}_{du}^{[4],tu}, \tag{21}$$

where v_6 can be on any place. Using 4-ary product (20) and the definition (21), we obtain the manifest form of the polyadic unit of the 4-ary algebra of 6-tuples $\mathbb{A}_{du}^{[4],tu}$

$$\mathsf{e}_6^{[4]} = \begin{pmatrix} \mathbf{1} \\ 0 \\ \mathbf{1} \\ 0 \\ \mathbf{1} \\ 0 \end{pmatrix} \in \mathbb{A}_{du}^{[4],tu}. \tag{22}$$

It follows from (20) that $\mathbb{A}_{du}^{[4],tu}$ (and so also $\mathbb{A}_{dual}^{[4]}$) is a non-division 4-ary algebra, and not a field (similarly to the ordinary binary $\mathbb{A}_{du}$), because it contains 4-ary idempotents and zero divisors, for instance,

$$\mu_{\mathsf{v}}^{[4]}\left[\begin{pmatrix} 0 \\ b_1 \\ 0 \\ b_2 \\ a_3 \\ b_3 \end{pmatrix}^4\right] = \mu_{\mathsf{v}}^{[4]}\left[\begin{pmatrix} a_1 \\ b_1 \\ 0 \\ b_2 \\ 0 \\ b_3 \end{pmatrix}^4\right] = \mu_{\mathsf{v}}^{[4]}\left[\begin{pmatrix} 0 \\ b_1 \\ a_2 \\ b_2 \\ 0 \\ b_3 \end{pmatrix}^4\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0_{\mathsf{v}}. \tag{23}$$

Thus, the application of the matrix polyadization procedure to the commutative, associative, unital, two-dimensional, non-division algebra of dual numbers $\mathbb{A}_{du}$ gives a noncommutative, 4-ary nonderived totally associative, unital, six-dimensional, non-division algebra $\mathbb{A}_{du}^{[4]}$ over $\mathbb{R}$, which we call the 4-ary dual numbers.

4. Polyadization of Division Algebras

Let us consider the matrix polyadization procedure for the division algebras $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ in more detail, paying attention to invertibility and norms.

Recall that the binary division algebra $\mathbb{D}$ (without zero element) forms a group (with respect to binary multiplication) having the inverse (10). The polyadic counterpart of the binary inverse is the querelement (5).

Theorem 1. *The nonderived n-ary algebra (13) constructed from the binary division algebra $\mathbb{D}$ is the n-ary algebra*

$$\mathbb{D}^{[n]} = \left\langle \left\{ \mathbf{Z}^{[n]} \right\} \mid (+), \boldsymbol{\mu}_{\mathbf{Z}}^{[n]}, \widetilde{\mathbf{Z}}^{[n]} \right\rangle, \tag{24}$$

where $\widetilde{\mathbf{Z}}^{[n]}$ is the querelement

$$\widetilde{\mathbf{Z}} = \widetilde{\mathbf{Z}}^{[n]} = \begin{pmatrix} 0 & \widetilde{z}_1 & \ldots & 0 & 0 \\ 0 & 0 & \widetilde{z}_2 & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \widetilde{z}_{n-2} \\ \widetilde{z}_{n-1} & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{25}$$

which satisfies (provided $\mathbb{D}$ is associative)

$$\boldsymbol{\mu}_{\mathbf{Z}}^{[n]} \left[\overbrace{\mathbf{Z}, \mathbf{Z} \ldots \mathbf{Z}}^{n-1} \widetilde{\mathbf{Z}} \right] = \overbrace{\mathbf{Z} \cdot \mathbf{Z} \cdot \ldots \cdot \mathbf{Z}}^{n-1} \cdot \widetilde{\mathbf{Z}} = \mathbf{Z}, \qquad \forall \mathbf{Z} \in \mathbb{D}^{[n]}, \tag{26}$$

and $\widetilde{\mathbf{Z}}$ can be on any place, such that

$$\widetilde{z}_i = z_{i-1}^{-1} z_{i-2}^{-1} \ldots z_2^{-1} z_1^{-1} z_{n-1}^{-1} z_{n-1}^{-1} \ldots z_{i+2}^{-1} z_{i+1}^{-1}, \quad z_i \in \mathbb{D} \backslash \{0\}. \tag{27}$$

Proof. The main relation (27) follows from (26) in components (15) as the following cycle products

$$\begin{aligned} \overbrace{z_1 z_2 \ldots z_{n-1} \widetilde{z}_1}^{n} &= z_1, \\ \overbrace{z_2 z_3 \ldots z_{n-1} z_1 \widetilde{z}_2}^{n} &= z_2, \\ &\vdots \\ \overbrace{z_{n-1} z_1 \ldots z_{n-2} \widetilde{z}_{n-1}}^{n} &= z_{n-1}, \qquad z_i, \widetilde{z}_i \in \mathbb{D} \backslash \{0\}, \end{aligned} \tag{28}$$

are obtained by applying z_i^{-1} (which exists in $\mathbb{D}$ for nonzero z (10)) from the left $(n-1)$ times (with suitable indices) to both sides of each equation in (28) to obtain $\widetilde{z}_i$. □

Corollary 1. *Each D-dimensional division algebra over the reals $\mathbb{D} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ (including $\mathbb{R}$ itself as the one-dimensional case) has as its n polyadic counterparts (where n is arbitrary) the nonderived n-ary non-division algebras $\mathbb{D}^{[n]}$ (24) of dimension $D(n-1)$ (having the polyadic unit (16) and the querelement (25) for invertible $z_i \in \mathbb{D} \backslash \{0\}$) constructed by the matrix polyadization procedure.*

Theorem 2. *The matrix polyadization procedure changes the invertibility properties of the initial algebra, that is, the polyadization of a binary division algebra $\mathbb{D}$ leads to a n-ary non-division algebra $\mathbb{D}^{[n]}$ for arbitrary n.*

Proof. The polyadization procedure is provided by monomial matrices which have a determinant proportional to the product of nonzero entries. The nonzero elements of $\mathbb{D}^{[n]}$ having some of $z_i = 0$ are noninvertible, and therefore, $\mathbb{D}^{[n]}$ is not a field. □

Nevertheless, a special subalgebra of $\mathbb{D}^{[n]}$ can be a division n-ary algebra, that is, when $\mathbf{Z}$-matrix is a monomial matrix, having one non-zero entry in each row and each column (see, e.g., [47]).

Theorem 3. *The elements of $\mathbb{D}^{[n]}$ which have all invertible $z_i \in \mathbb{D}\backslash\{0\}$ (or set of invertible Z-matrices $\det Z \neq 0$) form a subalgebra $\mathbb{D}^{[n]}_{\mathrm{div}} \subset \mathbb{D}^{[n]}$ which is the division n-ary algebra corresponding to the division algebra $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.*

The simplest case is obtained by the polyadization of the reals.

Example 2 (Five-ary real numbers)**.** *The 5-ary associative algebra of real numbers is $\mathbb{R}^{[5]} = \left\langle \left\{ \boldsymbol{R}^{[5]} \right\} \mid (+), \boldsymbol{\mu}^{[5]}_{\boldsymbol{R}} \right\rangle$, where $\boldsymbol{R}^{[5]}$ is the cyclic 4×4 block-shift matrix (11) with real entries*

$$\boldsymbol{R}^{[5]} = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_3 \\ a_4 & 0 & 0 & 0 \end{pmatrix}, \tag{29}$$

$$\det \boldsymbol{R}^{[5]} = a_1 a_2 a_3 a_4, \qquad a_i \in \mathbb{R}, \tag{30}$$

and the multiplication $\boldsymbol{\mu}^{[5]}_{\boldsymbol{R}}$ is an ordinary product of five matrices. Only with respect to the product of 5 elements $\boldsymbol{R}^{[5]}$ is the algebra closed, and therefore $\mathbb{R}^{[5]}$ is nonderived. In components, we have the braiding cyclic products (15) for a_i. If $a_i \in \mathbb{R} \setminus \{0\}$, the component equations for the querelement (28) (after the cancellation of nonzero a_i) become

$$a_2 a_3 a_4 \widetilde{a}_1 = 1, \tag{31}$$

$$a_3 a_4 a_1 \widetilde{a}_2 = 1, \tag{32}$$

$$a_4 a_1 a_2 \widetilde{a}_3 = 1, \tag{33}$$

$$a_1 a_2 a_3 \widetilde{a}_4 = 1. \tag{34}$$

Thus, the querelement for an invertible (29) is

$$\widetilde{\boldsymbol{R}}^{[5]} = \begin{pmatrix} 0 & \dfrac{1}{a_2 a_3 a_4} & 0 & 0 \\ 0 & 0 & \dfrac{1}{a_3 a_4 a_1} & 0 \\ 0 & 0 & 0 & \dfrac{1}{a_4 a_1 a_2} \\ \dfrac{1}{a_1 a_2 a_3} & 0 & 0 & 0 \end{pmatrix}, \quad a_i \in \mathbb{R} \setminus \{0\}, \tag{35}$$

and therefore, the algebra $\mathbb{R}^{[5]}$ of 5-ary real numbers is a non-division algebra, because the elements with some $a_i = 0$ are in $\mathbb{R}^{[5]}$, but they are noninvertible (due to (30)), and so $\mathbb{R}^{[5]}$ is not a field. But the subalgebra $\mathbb{R}^{[5]}_{\mathrm{div}} \subset \mathbb{R}^{[5]}$ of invertible (monomial) matrices $\boldsymbol{R}^{[5]}$ ($\det \boldsymbol{R}^{[5]} \neq 0$, with all $a_i \neq 0$) is a division n-ary algebra of reals (see Theorem 3).

Now, we provide the example of the 4-ary algebra of complex numbers, to compare it with the dual numbers of the same arity in Example 1.

Example 3 (4-ary complex numbers)**.** *First, we establish notations, as before. The commutative and associative two-dimensional algebra of complex numbers is $\mathbb{C} = \langle \{z\} \mid (+), (\cdot) \rangle$, where $z = a + b\boldsymbol{i}$ with $\boldsymbol{i}^2 = -1$, $a, b \in \mathbb{R}$. The binary multiplication $\mu^{[2]}_{\mathsf{v}} = \mu^{[2]}_{compl}$ of pairs*

$$\mathsf{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^{tu} = \left\langle \{\mathsf{v}_2\} \mid (+), \mu^{[2]}_{\mathsf{v}} \right\rangle \tag{36}$$

(where addition is component-wise, see Remark 1) is

$$\mu_{\mathsf{v}}^{[2]}\left[\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} a'' \\ b'' \end{pmatrix}\right] = \begin{pmatrix} a'a'' - b''b' \\ b''a' + b'a'' \end{pmatrix} \in \mathbb{C}^{tu}, \quad a', a'', b', b'' \in \mathbb{R}. \tag{37}$$

The algebra of pairs $\mathbb{C}^{tu}$ does not contain idempotents or zero divisors, its multiplication agrees with one of $\mathbb{C}$ ($\boldsymbol{z}' \cdot \boldsymbol{z}'' = \boldsymbol{z}$), and therefore, it is a division algebra, being isomorphic to $\mathbb{C}$

$$\mathbb{C}^{tu} \cong \mathbb{C}. \tag{38}$$

Using the matrix polyadization procedure, we construct the nonderived 4-ary algebra of complex numbers $\mathbb{C}^{[4]} = \left\langle \left\{ \boldsymbol{Z}^{[4]} \right\} \mid (+), \boldsymbol{\mu}_{\boldsymbol{Z}}^{[4]} \right\rangle$ of the dimension $D^{[4]} = 6$ (see (14)) by introducing the following 3×3 matrix (11)

$$\boldsymbol{Z} = \boldsymbol{Z}^{[4]} = \begin{pmatrix} 0 & \boldsymbol{z}_1 & 0 \\ 0 & 0 & \boldsymbol{z}_2 \\ \boldsymbol{z}_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 + b_1\boldsymbol{i} & 0 \\ 0 & 0 & a_2 + b_2\boldsymbol{i} \\ a_3 + b_3\boldsymbol{i} & 0 & 0 \end{pmatrix} \in \mathbb{C}^{[4]}, \quad \boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3 \in \mathbb{C}. \tag{39}$$

The 4-ary product of $\boldsymbol{Z}$-matrices (39) is

$$\boldsymbol{\mu}_{\boldsymbol{Z}}^{[4]}\left[\boldsymbol{Z}', \boldsymbol{Z}'', \boldsymbol{Z}''', \boldsymbol{Z}''''\right] = \boldsymbol{Z}'\boldsymbol{Z}''\boldsymbol{Z}'''\boldsymbol{Z}''''. \tag{40}$$

The corresponding to (40) cyclic product in the components (15) becomes

$$\begin{aligned} &\boldsymbol{z}_1'\boldsymbol{z}_2''\boldsymbol{z}_3'''\boldsymbol{z}_1'''' = \boldsymbol{z}_1, \\ &\boldsymbol{z}_2'\boldsymbol{z}_3''\boldsymbol{z}_1'''\boldsymbol{z}_2'''' = \boldsymbol{z}_2, \\ &\boldsymbol{z}_3'\boldsymbol{z}_1''\boldsymbol{z}_2'''\boldsymbol{z}_3'''' = \boldsymbol{z}_3, \qquad \boldsymbol{z}_i, \boldsymbol{z}_i', \boldsymbol{z}_i'', \boldsymbol{z}_i''', \boldsymbol{z}_i'''' \in \mathbb{C}. \end{aligned} \tag{41}$$

In terms of six-tuples v_6 over $\mathbb{R}$ (cf. (17)), the 4-ary multiplication $\mu_{\mathsf{v}}^{[4]}$ in $\mathbb{C}^{[4],tu} = \left\langle \{\mathsf{v}_6\} \mid (+), \mu_{\mathsf{v}}^{[4]} \right\rangle$ has the form (cf. (37))

$$\mu_{\mathsf{v}}^{[4]}\left[\begin{pmatrix} a_1' \\ b_1' \\ a_2' \\ b_2' \\ a_3' \\ b_3' \end{pmatrix}, \begin{pmatrix} a_1'' \\ b_1'' \\ a_2'' \\ b_2'' \\ a_3'' \\ b_3'' \end{pmatrix}, \begin{pmatrix} a_1''' \\ b_1''' \\ a_2''' \\ b_2''' \\ a_3''' \\ b_3''' \end{pmatrix}, \begin{pmatrix} a_1'''' \\ b_1'''' \\ a_2'''' \\ b_2'''' \\ a_3'''' \\ b_3'''' \end{pmatrix}\right]$$

$$= \begin{pmatrix} a_1'a_2''a_3'''a_1'''' - a_1'b_2''b_3'''a_1'''' - b_1'a_2''b_3'''a_1'''' - b_1'b_2''a_3'''a_1'''' - a_1'a_2''b_3'''b_1'''' - a_1'b_2''a_3'''b_1'''' - b_1'a_2''a_3'''b_1'''' + b_1'b_2''b_3'''b_1'''' \\ a_1'a_2''b_3'''a_1'''' + a_1'b_2''a_3'''a_1'''' + b_1'a_2''a_3'''a_1'''' + a_1'a_2''a_3'''b_1'''' - b_1'b_2''b_3'''a_1'''' - a_1'b_2''b_3'''b_1'''' - b_1'a_2''b_3'''b_1'''' - b_1'b_2''a_3'''b_1'''' \\ a_2'a_3''a_1'''a_2'''' - a_2'b_3''b_1'''a_2'''' - b_2'b_3''a_1'''a_2'''' - b_2'a_3''b_1'''a_2'''' - a_2'b_3''a_1'''b_2'''' - a_2'a_3''b_1'''b_2'''' - b_2'a_3''a_1'''b_2'''' + b_2'b_3''b_1'''b_2'''' \\ a_2'b_3''a_1'''a_2'''' + a_2'a_3''b_1'''a_2'''' + b_2'a_3''a_1'''a_2'''' + a_2'a_3''a_1'''b_2'''' - b_2'b_3''b_1'''a_2'''' - a_2'b_3''b_1'''b_2'''' - b_2'b_3''a_1'''b_2'''' - b_2'a_3''b_1'''b_2'''' \\ a_3'a_1''a_2'''a_3'''' - b_3'a_1''b_2'''a_3'''' - a_3'b_1''b_2'''a_3'''' - b_3'b_1''a_2'''a_3'''' - a_3'a_1''b_2'''b_3'''' - b_3'a_1''a_2'''b_3'''' - a_3'b_1''a_2'''b_3'''' + b_3'b_1''b_2'''b_3'''' \\ a_3'a_1''b_2'''a_3'''' + b_3'a_1''a_2'''a_3'''' + a_3'b_1''a_2'''a_3'''' + a_3'a_1''a_2'''b_3'''' - b_3'b_1''b_2'''a_3'''' - b_3'a_1''b_2'''b_3'''' - a_3'b_1''b_2'''b_3'''' - b_3'b_1''a_2'''b_3'''' \end{pmatrix}, \tag{42}$$

which is nonderived and noncommutative due to braidings in the cyclic product (19). The polyadic unit (4-ary unit) in the 4-ary algebra of 6-tuples is (see (2))

$$\mathsf{e}_6 = \begin{pmatrix} \mathbf{1} \\ 0 \\ \mathbf{1} \\ 0 \\ \mathbf{1} \\ 0 \end{pmatrix} \in \mathbb{C}^{[4],tu}, \tag{43}$$

$$\mu_{\mathsf{v}}^{[4]}[\mathsf{e}_6, \mathsf{e}_6, \mathsf{e}_6, \mathsf{v}_6] = \mu_{\mathsf{v}}^{[4]}[\mathsf{e}_6, \mathsf{e}_6, \mathsf{v}_6, \mathsf{e}_6] = \mu_{\mathsf{v}}^{[4]}[\mathsf{e}_6, \mathsf{v}_6, \mathsf{e}_6, \mathsf{e}_6] = \mu_{\mathsf{v}}^{[4]}[\mathsf{v}_6, \mathsf{e}_6, \mathsf{e}_6, \mathsf{e}_6] = \mathsf{v}_6. \tag{44}$$

The querelement (5) and (25) for invertible elements of the 4-ary algebra of complex numbers $\mathbb{C}^{[4]}$ *has the matrix form which follows from the equations (28)*

$$\widetilde{\boldsymbol{Z}}^{[4]} = \begin{pmatrix} 0 & \frac{1}{z_2 z_3} & 0 \\ 0 & 0 & \frac{1}{z_1 z_3} \\ \frac{1}{z_1 z_2} & 0 & 0 \end{pmatrix} \in \mathbb{C}^{[4]}, \quad \boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3 \in \mathbb{C} \backslash \{\boldsymbol{0}\}. \tag{45}$$

Thus, $\mathbb{C}^{[4]} = \left\langle \left\{ \boldsymbol{Z}^{[4]} \right\} \mid (+), \boldsymbol{\mu}^{[4]}, \widetilde{(\ )} \right\rangle$ *is the nonderived 4-ary non-division algebra over* $\mathbb{R}$ *obtained by the matrix polyadization procedure from the algebra* $\mathbb{C}$ *of complex numbers. The 4-ary algebra of complex numbers* $\mathbb{C}^{[4]}$ *is not a field, because it contains noninvertible nonzero elements (with some* $\boldsymbol{z}_i = \boldsymbol{0}$*).*

In $\mathbb{C}^{[4],tu}$ *the querelement is given by the following 6-tuple*

$$\widetilde{\mathrm{v}}_6 = \widetilde{\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \end{pmatrix}} = \begin{pmatrix} \frac{a_2 a_3 - b_2 b_3}{(a_2 b_3 + a_3 b_2)^2 + (a_2 a_3 - b_2 b_3)^2} \\ -\frac{a_2 b_3 + a_3 b_2}{(a_2 b_3 + a_3 b_2)^2 + (a_2 a_3 - b_2 b_3)^2} \\ \frac{a_1 a_3 - b_1 b_3}{(a_1 b_3 + a_3 b_1)^2 + (a_1 a_3 - b_1 b_3)^2} \\ -\frac{a_1 b_3 + a_3 b_1}{(a_1 b_3 + a_3 b_1)^2 + (a_1 a_3 - b_1 b_3)^2} \\ \frac{a_1 a_2 - b_1 b_2}{(a_1 b_2 + a_2 b_1)^2 + (a_1 a_2 - b_1 b_2)^2} \\ -\frac{a_1 b_2 + a_2 b_1}{(a_1 b_2 + a_2 b_1)^2 + (a_1 a_2 - b_1 b_2)^2} \end{pmatrix}, \quad a_i, b_i \in \mathbb{R}. \tag{46}$$

Therefore, $\mathbb{C}^{[4],tu}$ *is a division nonderived 4-ary algebra isomorphic to* $\mathbb{C}^{[4]}$*.*

Thus, the application of the matrix polyadization procedure to the commutative, associative, unital, two-dimensional, division algebra of complex numbers $\mathbb{C}$ *gives the noncommutative, 4-ary nonderived, totally associative, unital, six-dimensional, non-division algebra* $\mathbb{C}^{[4]}$ *over* $\mathbb{R}$ *(with the corresponding isomorphic 4-ary non-division algebra of 6-tuples* $\mathbb{C}^{[4],tu}$*), which we call the 4-ary complex numbers. The subalgebra* $\mathbb{C}^{[4]}_{\mathrm{div}} \subset \mathbb{C}^{[4]}$ *of invertible matrices* $\mathbf{Z}^{[4]}$ *(*$\det \mathbf{Z}^{[4]} \neq \mathbf{0}$*, with all* $\boldsymbol{z}_i \neq \mathbf{0}$*) is the division 4-ary algebra of complex numbers (by Theorem 3).*

We then consider the polyadization of the noncommutative quaternion algebra $\mathbb{H}$.

Example 4 (Ternary quaternions)**.** *The associative four-dimensional algebra of quaternions is given by*

$$\begin{aligned} &\mathbb{H} = \langle \{\boldsymbol{q}\} \mid (+), (\cdot) \rangle, \quad \boldsymbol{q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}, \\ &\boldsymbol{ij} = \boldsymbol{k}, \boldsymbol{ij} = -\boldsymbol{ji}, (+\text{cycled}), \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1, \ \ a, b, c, d \in \mathbb{R}. \end{aligned} \tag{47}$$

The binary multiplication $\mu_{\mathrm{v}}^{[4]}$ *of the quadruples*

$$\mathrm{v}_4 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{H}^{tu} = \left\langle \{\mathrm{v}_4\} \mid (+), \mu_{\mathrm{v}}^{[2]} \right\rangle \tag{48}$$

is (we present the binary product in our notation here for completeness to compare with the ternary case below)

$$\mu_{\mathsf{v}}^{[4]}\left[\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix}, \begin{pmatrix} a'' \\ b'' \\ c'' \\ d'' \end{pmatrix}\right] = \begin{pmatrix} a'a'' - b'b'' - c'c'' - d'd'' \\ a'b'' + b'a'' - d'c'' + c'd'' \\ a'c'' + c'a'' + d'b'' - b'd'' \\ a'd'' + b'c'' - c'b'' + d'a'' \end{pmatrix} \in \mathbb{H}^{tu}, \tag{49}$$

where $a', a'', b', b'', c', c'', d', d'' \in \mathbb{R}$. The binary algebra of quadruples $\mathbb{H}^{tu}$ is a noncommutative division algebra (isomorphic to $\mathbb{H}$), without idempotents or zero divisors.

By the above matrix polyadization procedure, we construct the nonderived ternary algebra of quaternions $\mathbb{H}^{[3]} = \left\langle \left\{ \boldsymbol{Q}^{[3]} \right\} \mid (+), \boldsymbol{\mu}_{\boldsymbol{Q}}^{[3]} \right\rangle$ of the dimension $D^{[3]} = 8$ (see (14)) by introducing the following 2×2 matrix (11)

$$\boldsymbol{Q}^{[3]} = \begin{pmatrix} 0 & \boldsymbol{q}_1 \\ \boldsymbol{q}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 + b_1\boldsymbol{i} + c_1\boldsymbol{j} + d_1\boldsymbol{k} \\ a_2 + b_2\boldsymbol{i} + c_2\boldsymbol{j} + d_2\boldsymbol{k} & 0 \end{pmatrix} \in \mathbb{H}^{[3]}, \quad \boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{H}. \tag{50}$$

The nonderived ternary product of $\boldsymbol{Q}$-matrices (39) is

$$\boldsymbol{\mu}_{\boldsymbol{Q}}^{[3]}\left[\boldsymbol{Q}', \boldsymbol{Q}'', \boldsymbol{Q}'''\right] = \boldsymbol{Q}'\boldsymbol{Q}''\boldsymbol{Q}'''. \tag{51}$$

The cyclic product corresponding to (40) in components (15) becomes

$$\begin{aligned} &\boldsymbol{q}_1'\boldsymbol{q}_2''\boldsymbol{q}_1''' = \boldsymbol{q}_1, \\ &\boldsymbol{q}_2'\boldsymbol{q}_1''\boldsymbol{q}_2''' = \boldsymbol{q}_2, \quad \boldsymbol{q}_i, \boldsymbol{q}_i', \boldsymbol{q}_i'', \boldsymbol{q}_i''' \in \mathbb{H}. \end{aligned} \tag{52}$$

In terms of eight-tuples v_8 over $\mathbb{R}$ (cf. (17)), the ternary multiplication $\mu_{\mathsf{v}}^{[3]}$ in $\mathbb{H}^{[3],tu} = \left\langle \{\mathsf{v}_8\} \mid (+), \mu_{\mathsf{v}}^{[3]} \right\rangle$ has the form (cf. (37))

$$\mu_{\mathsf{v}}^{[3]}\left[\mathsf{v}_8', \mathsf{v}_8'', \mathsf{v}_8'''\right] = \mu_{\mathsf{v}}^{[3]}\left[\begin{pmatrix} a_1' \\ b_1' \\ c_1' \\ d_1' \\ a_2' \\ b_2' \\ c_2' \\ d_2' \end{pmatrix}, \begin{pmatrix} a_1'' \\ b_1'' \\ c_1'' \\ d_1'' \\ a_2'' \\ b_2'' \\ c_2'' \\ d_2'' \end{pmatrix}, \begin{pmatrix} a_1''' \\ b_1''' \\ c_1''' \\ d_1''' \\ a_2''' \\ b_2''' \\ c_2''' \\ d_2''' \end{pmatrix}\right] =$$

$$\begin{pmatrix} (a_1'a_2'' - b_1'b_2'' - c_1'c_2'' - d_1'd_2'')a_1''' - (a_1'b_2'' + b_1'a_2'' + c_1'd_2'' - d_1'c_2'')b_1''' - (a_1'c_2'' + c_1'a_2'' + d_1'b_2'' - b_1'd_2'')c_1''' - (a_1'd_2'' + d_1'a_2'' + b_1'c_2'' - c_1'b_2'')d_1''' \\ (a_1'b_2'' + b_1'a_2'' + c_1'd_2'' - d_1'c_2'')a_1''' + (a_1'a_2'' - b_1'b_2'' - c_1'c_2'' - d_1'd_2'')b_1''' - (a_1'd_2'' + d_1'a_2'' + b_1'c_2'' - c_1'b_2'')c_1''' + (a_1'c_2'' + c_1'a_2'' + d_1'b_2'' - b_1'd_2'')d_1''' \\ (a_1'c_2'' + c_1'a_2'' + d_1'b_2'' - b_1'd_2'')a_1''' + (a_1'd_2'' + d_1'a_2'' + b_1'c_2'' - c_1'b_2'')b_1''' + (a_1'a_2'' - b_1'b_2'' - c_1'c_2'' - d_1'd_2'')c_1''' - (a_1'b_2'' + b_1'a_2'' + c_1'd_2'' - d_1'c_2'')d_1''' \\ (a_1'd_2'' + d_1'a_2'' + b_1'c_2'' - c_1'b_2'')a_1''' - (a_1'c_2'' + c_1'a_2'' + d_1'b_2'' - b_1'd_2'')b_1''' + (a_1'b_2'' + b_1'a_2'' + c_1'd_2'' - d_1'c_2'')c_1''' + (a_1'a_2'' - b_1'b_2'' - c_1'c_2'' - d_1'd_2'')d_1''' \\ (a_2'a_1'' - b_2'b_1'' - c_2'c_1'' - d_2'd_1'')a_2''' - (a_2'b_1'' + b_2'a_1'' + c_2'd_1'' - d_1'c_2'')b_2''' - (a_2'c_1'' + c_2'a_1'' + d_2'b_1'' - b_2'd_1'')c_2''' + (a_2'd_1'' + d_2'a_1'' + b_2'c_1'' - c_2'b_1'')d_2''' \\ (a_1'b_2'' + b_1'a_2'' + c_1'd_2'' - d_1'c_2'')a_2''' - (a_2'b_1'' + b_2'a_1'' + c_2'd_1'' - d_2'c_1'')b_2''' - (a_2'd_1'' + d_2'a_1'' + b_2'c_1'' - c_2'b_1'')c_2''' + (a_2'c_1'' + c_2'a_1'' + d_2'b_1'' - b_2'd_1'')d_2''' \\ (a_2'c_1'' + c_2'a_1'' + d_2'b_1'' - b_2'd_1'')a_2''' + (a_2'd_1'' + d_2'a_1'' + b_2'c_1'' - c_2'b_1'')b_2''' + (a_2'a_1'' - b_2'b_1'' - c_2'c_1'' - d_2'd_1'')c_2''' - (a_2'b_1'' + b_2'a_1'' + c_2'd_1'' - d_2'c_1'')d_2''' \\ (a_2'd_1'' + d_2'a_1'' + b_2'c_1'' - c_2'b_1'')a_2''' - (a_2'c_1'' + c_2'a_1'' + d_2'b_1'' - b_2'd_1'')b_2''' + (a_2'b_1'' + b_2'a_1'' + c_2'd_1'' - d_2'c_1'')c_2''' + (a_2'a_1'' - b_2'b_1'' - c_2'c_1'' - d_2'd_1'')d_2''' \end{pmatrix} \tag{53}$$

where $a_i', a_i'', a_i''', b_i', b_i'', b_i''', c_i', c_i'', c_i''', d_i', d_i'', d_i''' \in \mathbb{R}$.

Remark 3. *It is important to note that, although the ternary multiplication of 8-tuples (53) is nonderived and noncommutative, it is not the ordinary product of three quaternion pairs, but corresponds to the nontrivial cyclic braided products of (52).*

The polyadic unit (ternary unit) e_8 *in the ternary algebra of* 8*-tuples is (see (2))*

$$\mathsf{e}_8 = \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \\ 0 \\ \mathbf{1} \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{H}^{[3],tu}, \tag{54}$$

$$\mu_{\mathsf{v}}^{[3]}[\mathsf{e}_8, \mathsf{e}_8, \mathsf{v}_8] = \mu_{\mathsf{v}}^{[3]}[\mathsf{e}_8, \mathsf{v}_8, \mathsf{e}_8] = \mu_{\mathsf{v}}^{[3]}[\mathsf{v}_8, \mathsf{e}_8, \mathsf{e}_8] = \mathsf{v}_8. \tag{55}$$

The querelement (5) and (25) of the nonderived noncommutative 8*-dimensional ternary algebra of quaternions* $\mathbb{H}^{[3]}$ *has the matrix form which follows from the general equations (28)*

$$\widetilde{\mathbf{Q}}^{[3]} = \begin{pmatrix} 0 & \boldsymbol{q}_2^{-1} \\ \boldsymbol{q}_1^{-1} & 0 \end{pmatrix} \in \mathbb{H}^{[3]}, \quad \boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{H} \backslash \{0\}. \tag{56}$$

Therefore, $\mathbb{H}^{[3]} = \left\langle \left\{ \mathbf{Q}^{[3]} \right\} \mid (+), \boldsymbol{\mu}_{\mathbf{Q}}^{[3]}, \widetilde{(\ )} \right\rangle$ *is the nonderived noncommutative ternary non-division algebra obtained by the matrix polyadization procedure from the algebra* $\mathbb{H}$ *of quaternions. Because the elements* $\mathbf{Q}^{[3]}$ *with* $\boldsymbol{q}_1 = 0$ *or* $\boldsymbol{q}_2 = 0$ *are nonzero, but noninvertible,* $\mathbb{H}^{[3]}$ *is not a field.*

In $\mathbb{H}^{[3],tu}$ *the querelement is given by the following* 8*-tuple*

$$\widetilde{\mathsf{v}}_8 = \widetilde{\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}} = \begin{pmatrix} \frac{a_1}{a_1^2 + b_1^2 + c_1^2 + d_1^2} \\ -\frac{b_1}{a_1^2 + b_1^2 + c_1^2 + d_1^2} \\ -\frac{c_1}{a_1^2 + b_1^2 + c_1^2 + d_1^2} \\ -\frac{d_1}{a_1^2 + b_1^2 + c_1^2 + d_1^2} \\ \frac{a_2}{a_2^2 + b_2^2 + c_2^2 + d_2^2} \\ -\frac{b_2}{a_2^2 + b_2^2 + c_2^2 + d_2^2} \\ -\frac{c_2}{a_2^2 + b_2^2 + c_2^2 + d_2^2} \\ -\frac{d_2}{a_2^2 + b_2^2 + c_2^2 + d_2^2} \end{pmatrix}, \quad a_{1,2}^2 + b_{1,2}^2 + c_{1,2}^2 + d_{1,2}^2 \neq 0, \ a_i, b_i, c_i, d_i \in \mathbb{R}. \tag{57}$$

Therefore, $\mathbb{H}^{[3],tu}$ *is a non-division ternary algebra isomorphic to* $\mathbb{H}^{[3]}$*.*

To conclude, the application of the matrix polyadization procedure to the noncommutative, associative, unital, four-dimensional, division algebra of quaternions $\mathbb{H}$ *gives the noncommutative, nonderived ternary, totally associative, unital,* 8*-dimensional, non-division algebra* $\mathbb{H}^{[3]}$ *over* $\mathbb{R}$ *(with the corresponding isomorphic ternary non-division algebra of* 8*-tuples* $\mathbb{H}^{[3],tu}$*), which we call the ternary quaternions. The subalgebra* $\mathbb{H}^{[3]}_{\text{div}} \subset \mathbb{H}^{[3]}$ *of invertible matrices* $\mathbf{Q}^{[3]}$ (det $\mathbf{Q}^{[3]} \neq \mathbf{0}$*, with all* $\boldsymbol{q}_i \neq \mathbf{0}$*) is the division ternary algebra of quaternions (see Theorem 3).*

5. Polyadic Norms

The division algebras $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are normed as vector spaces, and the corresponding Euclidean 2-norm is multiplicative (9), such that the corresponding mapping is a binary homomorphism. It would be worthwhile to define a polyadic analog of the binary norm $\| \ \|$ having similar properties.

Definition 2. *We define the polyadic (n-ary) norm* $\| \ \|^{[n]}$ *for the n-ary algebra* $\mathbb{D}^{[n]}$*, that is obtained from* $\mathbb{D}$ *by the matrix polyadization procedure (11), as the product (in* $\mathbb{R}$*) of the component norms*

$$\|\mathbf{Z}\|^{[n]} = \|z_1\|\|z_2\| \ldots \|z_{n-1}\| \in \mathbb{R}, \quad \mathbf{Z} \in \mathbb{D}^{[n]}, \ z_i \in \mathbb{D}. \tag{58}$$

Corollary 2. *The polyadic norm (58) is zero for the noninvertible elements of* $\mathbb{D}^{[n]}$*, having some* $z_i = 0$.

Therefore, it is worthwhile to consider a polyadic norm for invertible elements of $\mathbb{D}^{[n]}$ only.

Proposition 4. *The division n-ary subalgebras* $\mathbb{D}^{[n]}_{\text{div}} \subset \mathbb{D}^{[n]}$ *are normed n-ary algebras with respect to the polyadic norm* $\| \ \|^{[n]}$ *(58).*

Let us consider some properties of the polyadic norm (58).

Proposition 5. *The polyadic norm* $\| \ \|^{[n]}$ *introduced above (58) has the following properties (for invertible* $\mathbf{Z} \in \mathbb{D}^{[n]}$*)*

$$\left\|\mathbf{E}^{[n]}\right\|^{[n]} = 1 \in \mathbb{R}, \quad \mathbf{E}^{[n]} \in \mathbb{D}^{[n]}, \tag{59}$$

$$\|\lambda \mathbf{Z}\|^{[n]} = |\lambda|^{n-1} \|\mathbf{Z}\|^{[n]}, \quad \lambda \in \mathbb{R}, \tag{60}$$

$$\|\mathbf{Z}' + \mathbf{Z}''\|^{[n]} \leq \|\mathbf{Z}\|^{[n]} + \|\mathbf{Z}\|^{[n]}, \ \mathbf{Z} \in \mathbb{D}^{[n]}, \tag{61}$$

where $\mathbf{E}^{[n]}$ *is the polyadic unit in the n-ary algebra* $\mathbb{D}^{[n]}$ *(16).*

Proof. The first property is obvious, the second one follows from the definition (58), and the linearity of the ordinary Euclidean norm in $\mathbb{D}$ (7). The polyadic triangle inequality (61) follows from the binary triangle inequality (8), because of the binary addition of $\mathbf{Z}$-matrices of the cyclic block-shift form (11). □

The norms satisfying (60) are called norms of higher degree, and they were investigated for the binary case in [48].

The most important property of any (binary) norm is its multiplicativity (9).

Theorem 4. *The polyadic norm* $\| \ \|^{[n]}$ *defined in (58) is n-ary multiplicative (such that the corresponding map* $\mathbb{D}^{[n]} \to \mathbb{R}$ *is an n-ary homomorphism)*

$$\left\| \mu^{[n]}_{\mathbf{Z}} \left[\overbrace{\mathbf{Z}', \mathbf{Z}'' \ldots \mathbf{Z}'''}^{n} \right] \right\|^{[n]} = \overbrace{\|\mathbf{Z}'\|^{[n]} \cdot \|\mathbf{Z}''\|^{[n]} \cdot \ldots \cdot \|\mathbf{Z}'''\|^{[n]}}^{n}, \quad \mathbf{Z}', \mathbf{Z}'', \mathbf{Z}''' \in \mathbb{D}^{[n]}_{\text{div}}. \tag{62}$$

Proof. Consider the component form (15) of each multiplier $\mathbf{Z}$ in (62), then use the definition (58) and commutativity (as they are in $\mathbb{R}$) and the multiplicativity (9) of the ordinary binary norms $\| \ \|$ to rearrange the products of norms from l.h.s. to r.h.s. in (62). That is,

$$\overbrace{\|z_1' z_2'' \ldots z_{n-1}''' z_1''''\| \cdot \|z_2' z_3'' \ldots z_1''' z_2''''\| \cdot \ldots \cdot \|z_{n-1}' z_1'' \ldots z_{n-2}''' z_{n-1}''''\|}^{n-1}$$

$$= \overbrace{(\|z_1'\|\|z_2'\| \ldots \|z_{n-1}'\|) \cdot (\|z_1''\|\|z_2''\| \ldots \|z_{n-1}''\|) \cdot \ldots \cdot (\|z_1'''\|\|z_2'''\| \ldots \|z_{n-1}'''\|) \cdot (\|z_1''''\|\|z_2''''\| \ldots \|z_{n-1}''''\|)}^{n}. \tag{63}$$

□

Remark 4. *The n-ary multiplicativity (62) of the polyadic norm introduced in (58) is independent of the concrete form of the binary norm* $\| \ \|$*, and only the multiplicativity of the latter is needed.*

Proposition 6. *The polyadic norm of the querelement in n-ary division subalgebra* $\mathbb{D}_{\mathrm{div}}^{[n]}$ *is*

$$\left\|\widetilde{\mathbf{Z}}\right\|^{[n]} = \frac{1}{\left(\|\mathbf{Z}\|^{[n]}\right)^{n-2}} \in \mathbb{R}_{>0}, \quad \forall \widetilde{\mathbf{Z}} \in \mathbb{D}_{\mathrm{div}}^{[n]}. \tag{64}$$

Proof. It follows from the component relations for the querelements $\|\widetilde{z}_i\|$ (28) and multiplicativity of the binary norm that

$$\begin{aligned}
\overbrace{\|z_1\|\|z_2\|\dots\|z_{n-1}\|\|\widetilde{z}_1\|}^{n} &= \|z_1\|, \\
\overbrace{\|z_2\|\|z_3\|\dots\|z_{n-1}\|\|z_1\|\|\widetilde{z}_2\|}^{n} &= \|z_2\|, \\
&\vdots \\
\overbrace{\|z_{n-1}\|\|z_1\|\dots\|z_{n-2}\|\|\widetilde{z}_{n-1}\|}^{n} &= \|z_{n-1}\|, \qquad z_i, \widetilde{z}_i \in \mathbb{D}, \ \ \|z_i\|, \|\widetilde{z}_i\| \in \mathbb{R}_{>0}.
\end{aligned} \tag{65}$$

Multiplying all the equations in (65) and using the definition of the polyadic norm (58), together with the component form (11) and the querelement (25), we obtain

$$\left(\|\mathbf{Z}\|^{[n]}\right)^{n-1} \left\|\widetilde{\mathbf{Z}}\right\|^{[n]} = \|\mathbf{Z}\|^{[n]}, \tag{66}$$

from which follows (64). □

Example 5. *In the division, the 4-ary algebra of complex numbers* $\mathbb{C}_{\mathrm{div}}^{[4]}$ *from Example 3 the polyadic norm becomes*

$$\|\mathbf{Z}\|^{[4]} = \sqrt{\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right)\left(a_3^2 + b_3^2\right)}, \quad a_i, b_i \in \mathbb{R}. \tag{67}$$

The polyadic norm of the querelement $\widetilde{\mathbf{Z}}$ *in* $\mathbb{C}^{[4]}$ *(64) is*

$$\left\|\widetilde{\mathbf{Z}}\right\|^{[4]} = \frac{1}{\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right)\left(a_3^2 + b_3^2\right)}, \qquad a_i^2 + b_i^2 \neq 0, \quad a_i, b_i \in \mathbb{R}. \tag{68}$$

Example 6. *In the division ternary algebra of quaternions* $\mathbb{H}_{\mathrm{div}}^{[3]}$ *from Example 4 the polyadic norm becomes*

$$\|\mathbf{Q}\|^{[3]} = \sqrt{\left(a_1^2 + b_1^2 + c_1^2 + d_1^2\right)\left(a_2^2 + b_2^2 + c_2^2 + d_2^2\right)}, \quad a_i, b_i \in \mathbb{R}. \tag{69}$$

The polyadic norm of the querelement $\widetilde{\mathbf{Q}}$ *in* $\mathbb{H}_{\mathrm{div}}^{[3]}$ *(64) is*

$$\left\|\widetilde{\mathbf{Q}}\right\|^{[3]} = \frac{1}{\sqrt{\left(a_1^2 + b_1^2 + c_1^2 + d_1^2\right)\left(a_2^2 + b_2^2 + c_2^2 + d_2^2\right)}}, \qquad a_i^2 + b_i^2 + c_i^2 + d_i^2 \neq 0, \quad a_i, b_i \in \mathbb{R}. \tag{70}$$

Further properties of the polyadic norm $\| \ \|^{[n]}$ can be investigated for invertible elements of concrete n-ary algebras.

6. Polyadic Analog of the Cayley–Dickson Construction

The standard method of obtaining the higher hypercomplex algebras is the Cayley–Dickson construction [18,49,50]. It is well known that all four binary division algebras $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ can be built in this way [8]. Here, we generalize the Cayley–Dickson construction to the polyadic (n-ary) division algebras introduced in the previous section. As a result, the number of polyadic division algebras becomes infinite (as opposed to just four in the binary case), because of the arbitrary initial and final arities of the algebras under consideration. For illustration, we present several low arity examples, since higher arity cases become too cumbersome and difficult to see. First, we recall in brief (just to install our notation) the ordinary (binary) Cayley–Dickson doubling process (in our notation, which is convenient for the polyadization procedure).

6.1. Abstract (Tuple) Approach

Consider the sequence of algebras $\mathbb{A}_\ell$, $\ell \geq 0$, over the reals, starting from $\mathbb{A}_0 = \mathbb{R}$. The main idea is to repeat the doubling process of the complex number construction using pairs (doubles) (37) and taking into account the isomorphism (38) at each stage $\mathbb{A}_\ell \to \mathbb{A}_{\ell+1}$. Let us denote the binary algebra over $\mathbb{R}$ on the ℓ-th stage with the underlying set $\{\mathbb{A}_\ell\}$ as

$$\mathbb{A}_\ell = \left\langle \{\mathbb{A}_\ell\} \mid (+), \mu_\ell^{[2]}, (*_\ell) \right\rangle, \tag{71}$$

where $(*_\ell)$ is involution in $\mathbb{A}_\ell$ and $\mu_\ell^{[2]} : \mathbb{A}_\ell \otimes \mathbb{A}_\ell \to \mathbb{A}_\ell$ is its binary multiplication, and we also will write $\mu_\ell^{[2]} \equiv (\cdot_\ell)$. The corresponding algebra of doubles (2-tuples)

$$\mathsf{v}_{2(\ell)} = \begin{pmatrix} \boldsymbol{a}_{(\ell)} \\ \boldsymbol{b}_{(\ell)} \end{pmatrix}, \quad \boldsymbol{a}_{(\ell)}, \boldsymbol{b}_{(\ell)} \in \mathbb{A}_\ell, \tag{72}$$

is denoted by

$$\mathbb{A}_\ell^{tu} = \left\langle \left\{ \mathsf{v}_{2(\ell)} \right\} \mid (+), \mu_{\mathsf{v}(\ell)}^{[2]}, (*_\ell^{tu}) \right\rangle, \tag{73}$$

where $(*_\ell^{tu})$ is involution in $\mathbb{A}_\ell^{tu}$ and $\mu_{\mathsf{v}(\ell)}^{[2]} : \mathbb{A}_\ell^{tu} \otimes \mathbb{A}_\ell^{tu} \to \mathbb{A}_\ell^{tu}$ is the binary product of doubles. If $\ell = 0$, then the conjugation is the identity map, as it should be for the reals $\mathbb{R}$. The addition and scalar multiplication are made componentwise in the standard way.

In this notation, the Cayley–Dickson doubling process is defined by the recurrent multiplication formula

$$\mu_{\mathsf{v}(\ell+1)}^{[2]} \left[\begin{pmatrix} \boldsymbol{a}'_{(\ell)} \\ \boldsymbol{b}'_{(\ell)} \end{pmatrix}, \begin{pmatrix} \boldsymbol{a}''_{(\ell)} \\ \boldsymbol{b}''_{(\ell)} \end{pmatrix} \right] = \begin{pmatrix} \mu_\ell^{[2]} \left[\boldsymbol{a}'_{(\ell)}, \boldsymbol{a}''_{(\ell)} \right] - \mu_\ell^{[2]} \left[\left(\boldsymbol{b}''_{(\ell)} \right)^{*_\ell}, \boldsymbol{b}'_{(\ell)} \right] \\ \mu_\ell^{[2]} \left[\boldsymbol{b}''_{(\ell)}, \boldsymbol{a}'_{(\ell)} \right] + \mu_\ell^{[2]} \left[\boldsymbol{b}'_{(\ell)}, \left(\boldsymbol{a}''_{(\ell)} \right)^{*_\ell} \right] \end{pmatrix} \tag{74}$$

$$= \begin{pmatrix} \boldsymbol{a}'_{(\ell)} \cdot_\ell \boldsymbol{a}''_{(\ell)} - \left(\boldsymbol{b}''_{(\ell)} \right)^{*_\ell} \cdot_\ell \boldsymbol{b}'_{(\ell)} \\ \boldsymbol{b}''_{(\ell)} \cdot_\ell \boldsymbol{a}'_{(\ell)} + \boldsymbol{b}'_{(\ell)} \cdot_\ell \left(\boldsymbol{a}''_{(\ell)} \right)^{*_\ell} \end{pmatrix} \equiv \begin{pmatrix} \boldsymbol{a}_{(\ell+1)} \\ \boldsymbol{b}_{(\ell+1)} \end{pmatrix} \in \mathbb{A}_{\ell+1}^{tu}, \tag{75}$$

and the recurrent conjugation

$$\begin{pmatrix} \boldsymbol{a}_{(\ell)} \\ \boldsymbol{b}_{(\ell)} \end{pmatrix}^{*_{\ell+1}^{tu}} = \begin{pmatrix} \boldsymbol{a}_{(\ell)}^{*_\ell} \\ -\boldsymbol{b}_{(\ell)} \end{pmatrix} \in \mathbb{A}_{\ell+1}^{tu}, \qquad \boldsymbol{a}_{(\ell)}, \boldsymbol{b}_{(\ell)} \in \mathbb{A}_\ell. \tag{76}$$

Then, we use the isomorphism (38) which now becomes

$$\mathbb{A}_{\ell+1}^{tu} \cong \mathbb{A}_{\ell+1}. \tag{77}$$

To go to the next level of recursion from that obtained so far, $\mathbb{A}_{\ell+1}$, we use (74)–(77) while changing $\ell \to \ell + 1$. The dimension of the algebra $\mathbb{A}_\ell$ is

$$D(\ell) = 2^\ell, \tag{78}$$

such that each element can be presented in the form of 2^ℓ-tuple of reals

$$\begin{pmatrix} a_{(\ell),1} \\ a_{(\ell),2} \\ \vdots \\ a_{(\ell),2^\ell} \end{pmatrix} \in \mathbb{A}_\ell, \tag{79}$$

and the conjugated 2^ℓ-tuple becomes

$$\begin{pmatrix} a_{(\ell),1} \\ -a_{(\ell)2} \\ \vdots \\ -a_{(\ell),2^\ell} \end{pmatrix}, \quad a_{(\ell),k} \in \mathbb{R}. \tag{80}$$

For clarity, we intentionally mark elements and operations at the ℓ-th level explicitly, because they really are different for different ℓ. The example with $\ell = 0$ just obtains the algebra of complex numbers $\mathbb{A}_1 \equiv \mathbb{C}$ from the algebra of reals $\mathbb{A}_0 \equiv \mathbb{R}$ (36)–(38). All the binary division algebras $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ can be obtained by the Cayley–Dickson construction [8].

6.2. Concrete (Hyperembedding) Approach

Alternatively, one can reparametrize the pairs (72) satisfying the complex-like multiplication (74), as the field extension $\widehat{\mathbb{A}}_\ell = \mathbb{A}_\ell(\boldsymbol{i}_\ell)$ by one complex-like unit $\boldsymbol{i}_\ell$ on each ℓ-th stage of iteration, $\ell \geq 0$. The Cayley–Dickson doubling process is given by the iterations

$$\widehat{\mathbb{A}}_{\ell+1} = \left\langle \{\mathbb{A}_\ell(\boldsymbol{i}_\ell)\} \mid (+), \widehat{\mu}^{[2]}_{\ell+1}, (*_{\ell+1}) \right\rangle (\boldsymbol{i}_{\ell+1}), \quad \boldsymbol{i}^2_\ell = -1, \quad \boldsymbol{i}^2_{\ell+1} = -1, \quad \boldsymbol{i}_{\ell+1}\boldsymbol{i}_\ell = -\boldsymbol{i}_\ell\boldsymbol{i}_{\ell+1}. \tag{81}$$

In a component form, it is given by

$$\boldsymbol{z}_{(\ell+1)} = \boldsymbol{z}_{(\ell),1} + \boldsymbol{z}_{(\ell),2} \cdot_{\ell+1} \boldsymbol{i}_{\ell+1} = \boldsymbol{z}_{(\ell),1} + \widehat{\mu}^{[2]}_{\ell+1}\left[\boldsymbol{z}_{(\ell),2}, \boldsymbol{i}_{\ell+1}\right] \in \widehat{\mathbb{A}}_{\ell+1}, \quad \boldsymbol{z}_{(\ell),1}, \boldsymbol{z}_{(\ell),2} \in \widehat{\mathbb{A}}_\ell. \tag{82}$$

The product in $\widehat{\mathbb{A}}_{\ell+1}$ can be expressed through the product of the previous stage from $\widehat{\mathbb{A}}_\ell$ and conjugation by using the anticommutation of the imaginary units from different stages $\boldsymbol{i}_{\ell+1}\boldsymbol{i}_\ell = -\boldsymbol{i}_\ell\boldsymbol{i}_{\ell+1}$ (see (81)). Thus, we obtain the standard complex-like multiplication (see (37) and (75)) on each ℓ-th stage of the Cayley–Dickson doubling process

$$\begin{aligned} &\boldsymbol{z}'_{(\ell+1)} \cdot_{\ell+1} \boldsymbol{z}''_{(\ell+1)} \\ &= \left(\boldsymbol{z}'_{(\ell),1} \cdot_\ell \boldsymbol{z}''_{(\ell),1} - \left(\boldsymbol{z}''_{(\ell),2}\right)^{*_\ell} \cdot_\ell \boldsymbol{z}'_{(\ell),2}\right) + \left(\boldsymbol{z}''_{(\ell),2} \cdot_\ell \boldsymbol{z}'_{(\ell),1} + \boldsymbol{z}'_{(\ell),2} \cdot_\ell \left(\boldsymbol{z}''_{(\ell),1}\right)^{*_\ell}\right) \cdot_{\ell+1} \boldsymbol{i}_{\ell+1}, \end{aligned} \tag{83}$$

or with the manifest form of different stage multiplications $\widehat{\mu}^{[2]}_\ell$ and $\widehat{\mu}^{[2]}_{\ell+1}$ (needed to go on higher arities)

$$\begin{aligned} \widehat{\mu}^{[2]}_{\ell+1}\left[\boldsymbol{z}'_{(\ell+1)}, \boldsymbol{z}''_{(\ell+1)}\right] &= \left(\widehat{\mu}^{[2]}_\ell\left[\boldsymbol{z}'_{(\ell),1}, \boldsymbol{z}''_{(\ell),1}\right] - \widehat{\mu}^{[2]}_\ell\left[\left(\boldsymbol{z}''_{(\ell),2}\right)^{*_\ell}, \boldsymbol{z}'_{(\ell),2}\right]\right) \\ &+ \widehat{\mu}^{[2]}_{\ell+1}\left[\left(\widehat{\mu}^{[2]}_\ell\left[\boldsymbol{z}''_{(\ell),2}, \boldsymbol{z}'_{(\ell),1}\right] + \widehat{\mu}^{[2]}_\ell\left[\boldsymbol{z}'_{(\ell),2}, \left(\boldsymbol{z}''_{(\ell),1}\right)^{*_\ell}\right]\right), \boldsymbol{i}_{\ell+1}\right]. \end{aligned} \tag{84}$$

Let us consider the example of quaternion construction in our notation.

Example 7. *In the case of quaternions $\ell = 1$ and $\widehat{\mathbb{A}}_{\ell+1} = \widehat{\mathbb{A}}_2 = \mathbb{H}$, while $\widehat{\mathbb{A}}_\ell = \widehat{\mathbb{A}}_1 = \mathbb{C}$. The recurrent relations (82) for $\ell = 0, 1$ become*

$$\boldsymbol{z}_{(2)} = \boldsymbol{z}_{(1),1} + \boldsymbol{z}_{(1),2} \cdot_2 \boldsymbol{i}_2, \quad \boldsymbol{z}_{(1),1}, \boldsymbol{z}_{(1),2} \in \widehat{\mathbb{A}}_1 = \mathbb{C}, \ \boldsymbol{z}_{(2)}, \boldsymbol{i}_2 \in \mathbb{H}, \tag{85}$$

$$\boldsymbol{z}_{(1),1} = \boldsymbol{z}_{(0),11} + \boldsymbol{z}_{(0),12} \cdot_1 \boldsymbol{i}_1, \quad \boldsymbol{z}_{(0),11}, \boldsymbol{z}_{(0),12} \in \widehat{\mathbb{A}}_0 = \mathbb{R}, \ \boldsymbol{z}_{(1),1}, \boldsymbol{i}_1 \in \mathbb{C}, \tag{86}$$

$$\boldsymbol{z}_{(1),2} = \boldsymbol{z}_{(0),21} + \boldsymbol{z}_{(0),22} \cdot_1 \boldsymbol{i}_1, \quad \boldsymbol{z}_{(0),21}, \boldsymbol{z}_{(0),22} \in \widehat{\mathbb{A}}_0 = \mathbb{R}, \ \boldsymbol{z}_{(1),2}, \boldsymbol{i}_1 \in \mathbb{C}. \tag{87}$$

After the substitution of (86) and (87) into (85), we obtain the expression of quaternions with real coefficients $z_{(0),\alpha\beta} \in \mathbb{R}$, $\alpha, \beta = 1, 2$, and two imagery units (from different parts of $\mathbb{H}$) $i_1 \in \mathbb{C} \setminus \mathbb{R} \subset \mathbb{H}$ and $i_2 \in \mathbb{H} \setminus \mathbb{C}$

$$\begin{aligned} z_{(2)} &= z_{(1),1} + z_{(1),2} \cdot_2 i_2 = \left(z_{(0),11} + z_{(0),12} \cdot_1 i_1 \right) + \left(z_{(0),21} + z_{(0),22} \cdot_1 i_1 \right) \cdot_2 i_2 \\ &= z_{(0),11} + z_{(0),12} \cdot_1 i_1 + z_{(0),21} \cdot_2 i_2 + \left(z_{(0),22} \cdot_1 i_1 \right) \cdot_2 i_2. \end{aligned} \tag{88}$$

To return to the standard notation (47), we put $z_{(2)} = q$, $z_{(0),11} = a$, $z_{(0),12} = b$, $z_{(0),21} = c$, $z_{(0),22} = d$, $i_1 = i \in \mathbb{C}$, $i_2 = j \in \mathbb{H} \setminus \mathbb{C}$, $i_1 \cdot_2 i_2 = k \in \mathbb{H}$ and obtain $q = a + bi + cj + dk$. Similarly, the complex-like multiplication (83) can be also applied twice with $\ell = 0, 1$ to obtain the quaternion multiplication in terms of real coefficients (49).

6.3. Polyadic Cayley–Dickson Process

Now, we provide a generalization of the Cayley-Dickson construction to the polyadic case in the framework of the second (embedding) approach using the field extension formalism (see Section 6.2). The main iteration relation (81) will now contain, instead of binary hypercomplex algebras $\widehat{\mathbb{A}}_\ell$, n-ary algebras $\widehat{\mathbb{A}}_\ell^{[n_\ell]}$ at each stage.

Definition 3. *The polyadic Cayley–Dickson process is defined as the following iteration on n-ary algebras*

$$\begin{aligned} \widehat{\mathbb{A}}_{\ell+1}^{[n_{\ell+1}]} &= \left\langle \left\{ \mathbb{A}_\ell^{[n_\ell]}(i_\ell) \right\} \mid (+), \widehat{\mu}_{\ell+1}^{[n_{\ell+1}]} \right\rangle (i_{\ell+1}), \\ & i_\ell^2 = -1, \quad i_{\ell+1}^2 = -1, \quad i_{\ell+1} i_\ell = -i_\ell i_{\ell+1}, \quad i_\ell \in \mathbb{A}_{\ell+1}^{[n_{\ell+1}]}, \quad n_\ell \geq 2, \quad \ell \geq 0. \end{aligned} \tag{89}$$

The concrete representation of the ℓ-th stage n-ary algebras $\widehat{\mathbb{A}}_\ell^{[n_\ell]}$ is not important for the general recurrence formula (89). Nevertheless, here we will use matrix polyadization to obtain higher n-ary non-division algebras (Section 3). First, we will need the obvious

Lemma 1. *The embedding of a block-monomial matrix into a block-monomial matrix gives a block-monomial matrix.*

Corollary 3. *If the binary Cayley–Dickson construction gives a division algebra, then the corresponding polyadic Cayley–Dickson process gives a nonderived n-ary non-division algebra, because of its noninvertible elements.*

Thus, the structure of the general algebra obtained by the polyadic Cayley–Dickson process is richer than a one block-shift monomial matrix (11), it is the "tower" of such monomial matrices of size $(n_\ell - 1) \times (n_\ell - 1)$ on ℓ-th stage embedded one into another. The connection between near arities (and matrix sizes) is

$$n_{\ell+1} = \varkappa (n_\ell - 1) + 1, \tag{90}$$

where $\varkappa$ is the polyadic power (1) [34].

Proposition 7. *If the number of stages of the polyadic Cayley–Dickson process is ℓ, then the dimension of the final algebra is (cf. (78))*

$$D_{CD}(\ell) = D(\mathbb{A}_{CD}^{[n_0, n_1, \ldots, n_\ell]}) = 2^\ell (n_0 - 1)(n_1 - 1) \ldots (n_\ell - 1) \tag{91}$$

where n_i are the arities of the intermediate algebras. The size of the final matrix becomes

$$(n_0 - 1)(n_1 - 1) \ldots (n_\ell - 1) \times (n_0 - 1)(n_1 - 1) \ldots (n_\ell - 1). \tag{92}$$

Proof. On the ℓ-th stage of the polyadic Cayley–Dickson process, each block-monomial $(n_\ell - 1) \times (n_\ell - 1)$ matrix has $(n_\ell - 1)$ nonzero blocks of the cycle-shift shape (11). The 0-th stage corresponds to reals, which gives $2^0(n_0 - 1)$ parameters. Then, the simple field extension $\ell = 1$ (82) with blocks of reals gives $(n_0 - 1) \cdot 2(n_1 - 1)$ parameters and so on. Thus, the ℓ-th stage is given by (91). □

More concretely, for the nonderived n-ary algebras, we have the dimensions

$$\begin{array}{ll} \mathbb{R}^{[n_0]}, & D(\mathbb{R}^{[n_0]}) = (n_0 - 1), \\ \mathbb{C}_{CD}^{[n_0,n_1]} = \mathbb{R}_{CD}^{[n_0]}(\boldsymbol{i}_1), & D(\mathbb{C}_{CD}^{[n_0,n_1]}) = 2(n_0 - 1)(n_1 - 1), \\ \mathbb{H}_{CD}^{[n_0,n_1,n_2]} = \mathbb{C}_{CD}^{[n_0,n_1]}(\boldsymbol{i}_2), & D(\mathbb{H}_{CD}^{[n_0,n_1,n_2]}) = 2^2(n_0 - 1)(n_1 - 1)(n_2 - 1), \\ \mathbb{O}_{CD}^{[n_0,n_1,n_2,n_3]} = \mathbb{H}_{CD}^{[n_0,n_1,n_2]}(\boldsymbol{i}_3), & D(\mathbb{O}_{CD}^{[n_0,n_1,n_2,n_3]}) = 2^3(n_0 - 1)(n_1 - 1)(n_2 - 1)(n_3 - 1), \\ \vdots & \vdots \\ \mathbb{A}_{CD,\ell}^{[n_0,n_1,\ldots,n_\ell]} = \mathbb{A}_{CD}^{[n_0,n_1,\ldots,n_{\ell-1}]}(\boldsymbol{i}_\ell), & D(\mathbb{A}_{CD}^{[n_0,n_1,\ldots,n_\ell]}) = 2^\ell(n_0 - 1)(n_1 - 1)\ldots(n_\ell - 1). \end{array} \tag{93}$$

The starting algebra $\mathbb{R}^{[n_0]}$ is presented by the $\mathbf{Z}$-matrix (11) with $n = n_0$ and real entries ($\mathbb{A} = \mathbb{R}$).

Definition 4. *The sequence of embedded cyclic shift block matrices in (93) will be called a polyadic block-shift tower.*

Remark 5. *The shapes of the final matrices (93) are different from the cyclic shift weighted matrices (11), but nevertheless, all the intermediate blocks are cyclic shift block matrices. Adjacent (in ℓ) matrices do not have an arbitrary size, but are connected by (90).*

Example 8. *To clarify the above general formulas, we provide the shape of polyadic quaternion matrices with $\ell = 2$, and $n_0 = 5$, $n_1 = 3$, $n_2 = 4$, as*

$$\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{pmatrix} \tag{94}$$

We have a 24×24 *monomial matrix,* $24 = (5-1)(3-1)(4-1)$ *by (93), where dots denote zeroes and stars denote nonzero entries. The dimension of this polyadic quaternion algebra represented by (94) is* $2^2 \cdot 24 = 96$*, because of the two simple field extensions (85)–(88). This is a* 13*-ary quaternion non-division algebra. If there were to be no further stages of the Cayley–Dickson process, and the* 24×24 *monomial matrix would not be composed, having the form (11), then it would give a* 25*-ary algebra.*

Thus, we present the polyadic Cayley-Dickson process in the component form (82), but instead of the elements $\boldsymbol{z}_{(\ell)}$ of the ℓ-th stage, we place the $\boldsymbol{Z}$-matrices (11) of suitable sizes. The resulting matrix is still monomial, and so its determinant is proportional to the product of elements. The subalgebra of invertible matrices corresponds to a division n-ary algebra.

Theorem 5. *The invertible elements (which are described by* $\boldsymbol{Z}$*-matrices with a nonzero determinant) of the polyadic Cayley–Dickson construction* $\mathbb{D}_{CD}^{[n]}$ *(for the first stages* $\ell = 0,1,2,3$*) (93) form the subalgebra* $\mathbb{D}_{CD,\mathrm{div}}^{[n]} \subset \mathbb{D}_{CD}^{[n]}$ *which is a polyadic division n-ary algebra* $\mathbb{D}_{CD,\mathrm{div}}^{[n]}$ *corresponding to the binary division algebras* $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$*.*

To conclude, the matrix polyadization procedure applied to division algebras leads, in general, to non-division algebras, because it is presented by monomial matrices (representing nonzero noninvertible elements) which become noninvertible when at least one entry vanishes. However, the subalgebras of invertible elements can be considered as new n-ary division algebras.

7. Polyadic Product of Vectors

Now, we will show that the matrix polyadization procedure (Section 3) is connected with a new product of vectors in a vector space, which we will consider below.

First, we recall some properties of monomial and related matrices [47]. An arbitrary monomial (or generalized permutation) matrix $\boldsymbol{M}_{mon}$ (over a field $\mathbb{F}^{\times} = \mathbb{F} \setminus \{0\}$, and so having nonzero entries) can be presented as a product of an invertible diagonal matrix $\boldsymbol{M}_{diag}$ and a permutation matrix $\boldsymbol{M}_{per}$

$$\boldsymbol{M}_{mon} = \boldsymbol{M}_{diag} \boldsymbol{M}_{per}. \tag{95}$$

The set of all $m \times m$ monomial matrices form a binary subgroup $\mathcal{G}_{mon}$ of the general linear group $GL_m(\mathbb{F})$, while the set of monomial matrices of the special fixed shape of the cyclic shift weighted matrices (11), form a nonderived $(m+1)$-ary group (see Section 3). Abstractly, (95) can be considered as the matrix representation of the wreath product of $\mathbb{F}^{\times}$ and the symmetric group S_m, because the group of diagonal matrices is isomorphic to $(\mathbb{F}^{\times})^m$ [47]. In general, a monomial representation of a binary group $\mathcal{G}$ is a homomorphism to some binary subgroup of $\mathcal{G}_{mon}$.

On the other hand, there exists the procedure of vectorization (see, e.g., [51,52]) which establishes a correspondence between (for instance square) $m \times m$ matrices and m^2-tuples (which can be interpreted as coordinate expressions of $m^2 \times 1$ column vectors in a vector space) derived by stacking the columns of the matrix, such that

$$\mathrm{vec}_{\mathbb{F}} : \mathbb{F}^{m \times m} \to \mathbb{F}^{m^2}. \tag{96}$$

In general, vectorization is a homomorphism. In particular, for diagonal matrices we have

$$\mathrm{vec}_{\mathbb{F}}\left(\boldsymbol{M}'_{diag} \boldsymbol{M}''_{diag}\right) = \mathrm{vec}_{\mathbb{F}}\left(\boldsymbol{M}'_{diag}\right) \odot \mathrm{vec}_{\mathbb{F}}\left(\boldsymbol{M}''_{diag}\right), \tag{97}$$

where $\odot$ is the element-wise product of m^2-tuples. For the monomial matrices (95), we define the following modification of vectorization.

Definition 5. *Reduced vectorization is the mapping of a monomial matrix into m-tuple*

$$\mathrm{vec}^{\times}_{\mathbb{F}} : \mathbb{F}^{m\times m} \to \mathbb{F}^{m}, \tag{98}$$

which is derived from vectorization (96) by omitting zeroes on both sides.

Remark 6. *The relation (97) is not valid for general monomial matrices (95). Nevertheless, we will show that, for a special class of monomial matrices, the cycled shift weighted matrices (11), the homomorphism (97) can be obtained for non-binary products on both sides, and that will allow us to define a new kind of product of vectors (in any vector space).*

Let $\mathcal{V}_{\mathbb{F}}$ be a finite-dimensional vector space (over a field $\mathbb{F}$) of dimension $m \geq 2$, such that

$$\mathcal{V}_{\mathbb{F}} \ni \mathbf{v} = \mathbf{v}(x) = \sum_{i=1}^{m} x_i \mathbf{e}_i, \quad x_i \in \mathbb{F}, \tag{99}$$

where $\mathbf{e}_i$ are canonical basis vectors.

In $\mathcal{V}_{\mathbb{F}}$ (satisfying the standard axioms), the only closed binary operation between vectors themselves is addition. Introducing another closed binary operation between vectors, which satisfies two-sided distributivity with the addition and compatibility with scalars (from $\mathbb{F}$), gives the bilinear product. A vector space $\mathcal{V}_{\mathbb{F}}$ (we do not consider its concrete realization) endowed with a bilinear product becomes an algebra over a field $\mathbb{F}$ (for further details and review, see, e.g., [53,54]).

The reduced vectorization (98) is not a binary homomorphism (see Remark 6), but can be a polyadic homomorphism, if we consider a special kind of monomial $m \times m$ matrices, the cyclic shift weighted matrices $\boldsymbol{M}_{shf}$ of the fixed shape (11).

To show this, we introduce a new product of vectors in a vector space $\mathcal{V}_{\mathbb{F}}$.

Definition 6. *In m-dimensional vector space $\mathcal{V}_{\mathbb{F}}$, we define $(m+1)$-ary (polyadic) product of vectors*

$$\boldsymbol{\mu}_{\star}^{[m+1]}\left[\mathbf{v}(x'), \mathbf{v}(x''), \ldots, \mathbf{v}(x''')\mathbf{v}(x'''')\right] = \left(\overbrace{x'_1 x''_2 \ldots x'''_m x''''_1}^{m+1}\right)\mathbf{e}_1$$
$$+ \left(\overbrace{x'_2 x''_3 \ldots x'''_1 x''''_2}^{m+1}\right)\mathbf{e}_2 + \ldots + \left(\overbrace{x'_m x''_1 \ldots x'''_{m-1} x''''_m}^{m+1}\right)\mathbf{e}_m, \quad x'_i, x''_i, \ldots, x'''_i x''''_i \in \mathbb{F}. \tag{100}$$

Remark 7. *The polyadic product of vectors (100) in components is not elementwise, but cyclic braided-like. Such products appeared in higher regular semigroups and braid groups [34], as well as in semisupermanifold theory [55].*

Consider the reduced vectorization $\mathrm{vec}^{\times}_{\mathbb{F}}$ (98) of the cyclic shift weighted matrices given by $\boldsymbol{M}_{shf}(x) \mapsto \mathbf{v}(x)$ or (cf. (11))

$$\boldsymbol{M}_{shf}(x) = \begin{pmatrix} 0 & x_1 & \ldots & 0 & 0 \\ 0 & 0 & x_2 & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & x_{m-1} \\ x_m & 0 & \ldots & 0 & 0 \end{pmatrix}$$
$$\mapsto \mathbf{v}(x) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_{m-1}\mathbf{e}_1 + x_m\mathbf{e}_m, \; x_i \in \mathbb{F}. \tag{101}$$

Proposition 8. *The reduced vectorization* $\mathrm{vec}_{\mathbb{F}}^{\times}$ *of the cyclic shift weighted matrices (101) is a polyadic or* $(m+1)$*-ary homomorphism.*

Proof. The product of $(m+1)$ matrices $\boldsymbol{M}_{shf}(x)$ (but not fewer) is a cyclic shift weighted matrix of the form such that the set $\left\{\boldsymbol{M}_{shf}(x)\right\}$ is a nonderived $(m+1)$-ary semigroup (cf. (15))

$$\overbrace{\boldsymbol{M}_{shf}(x')\boldsymbol{M}_{shf}(x'')\dots\boldsymbol{M}_{shf}(x''')\boldsymbol{M}_{shf}(x'''')}^{m+1}$$
$$=\begin{pmatrix} 0 & \overbrace{x'_1x''_2\dots x'''_mx''''_1}^{m+1} & \dots & 0 & 0 \\ 0 & 0 & \overbrace{x'_2x''_3\dots x'''_1x''''_2}^{m+1} & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \overbrace{x'_{m-1}x''_m\dots x'''_{m-2}x''''_{m-1}}^{m+1} \\ \overbrace{x'_mx''_1\dots x'''_{m-1}x''''_m}^{m+1} & 0 & \dots & 0 & 0 \end{pmatrix} \tag{102}$$
$$\mapsto \boldsymbol{\mu}_{\star}^{[m+1]}\left[\mathbf{v}(x'),\mathbf{v}(x''),\dots,\mathbf{v}(x''')\mathbf{v}(x'''')\right].$$

The statement now follows from comparing (102) with (100) and (101). □

Definition 7. *A vector space* $\mathcal{V}_{\mathbb{F}}$ *equipped with the* $(m+1)$*-ary (polyadic) product of vectors (100) becomes* $(m+1)$*-ary algebra* $\mathcal{A}^{[m+1]}(\mathbb{F})$ *over a field* $\mathbb{F}$*.*

Proposition 9. *The* $(m+1)$*-ary algebra* $\mathcal{A}^{[m+1]}(\mathbb{F})$ *is totally (polyadic) associative.*

Proof. It follows from the associativity of binary matrix multiplication $\boldsymbol{M}_{shf}(x)$ (102) and the definition of reduced vectorization (101). □

We define the structure constants of $\mathcal{A}^{[m+1]}(\mathbb{F})$ through the basis vectors $\mathbf{e}_i$ by

$$\boldsymbol{\mu}_{\star}^{[m+1]}\left[\mathbf{e}_{i_1},\mathbf{e}_{i_2},\dots,\mathbf{e}_{i_m},\mathbf{e}_{i_{m+1}}\right]=\sum_{j=1}^{m} f^{\,j}_{i_1,i_2,\dots,i_m,i_{m+1}}\mathbf{e}_j,\qquad f^{\,j}_{i_1,i_2,\dots,i_m,i_{m+1}}\in\mathbb{F}. \tag{103}$$

Theorem 6. *The structure constants of the* $(m+1)$*-ary algebra* $\mathcal{A}^{[m+1]}(\mathbb{F})$ *are*

$$f^{\,1}_{1,2,3,\dots,m-1,m,1}=1,\ f^{\,2}_{2,3,4,\dots m,1,2}=1,\dots,f^{\,m-1}_{m-1,m,1,2\dots,m-3,m-2,m-1}=1,\ f^{\,m}_{m,1,2\dots m-2,m-1,m}=1, \tag{104}$$

while $f^{\,j}_{i_1,i_2,\dots,i_m,i_{m+1}}$ *having other combinations of indices are zero.*

Proof. It follows from the explicit form of the $(m+1)$-ary product (100) and from the multilinearity of $\boldsymbol{\mu}_{\star}^{[m+1]}$, that is, the polyadic distributivity with addition (on each place)

$$\boldsymbol{\mu}_{\star}^{[m+1]}\left[\mathbf{v}_1(x')+\mathbf{v}_2(x'),\mathbf{v}(x''),\dots,\mathbf{v}(x'''),\mathbf{v}(x'''')\right]$$
$$=\boldsymbol{\mu}_{\star}^{[m+1]}\left[\mathbf{v}_1(x'),\mathbf{v}(x''),\dots,\mathbf{v}(x'''),\mathbf{v}(x'''')\right]+\boldsymbol{\mu}_{\star}^{[m+1]}\left[\mathbf{v}_2(x'),\mathbf{v}(x''),\dots,\mathbf{v}(x'''),\mathbf{v}(x'''')\right], \tag{105}$$

and compatibility with scalars

$$\begin{aligned}&\boldsymbol{\mu}_{\star}^{[m+1]}\left[\lambda'\mathbf{v}(x'),\lambda''\mathbf{v}(x''),\ldots,\lambda'''\mathbf{v}(x'''),\lambda''''\mathbf{v}(x'''')\right]\\&=(\lambda'\lambda''\ldots\lambda'''\lambda'''')\boldsymbol{\mu}_{\star}^{[m+1]}\left[\mathbf{v}_1(x'),\mathbf{v}(x''),\ldots,\mathbf{v}(x'''),\mathbf{v}(x'''')\right],\quad \lambda',\lambda'',\ldots\lambda''',\lambda''''\in\mathbb{F}.\end{aligned} \tag{106}$$

We insert the expansions (99) into (100) and then use the definition of the structure constants (103) taking into account (105) and (106) to obtain the result. □

Corollary 4. *If $\boldsymbol{M}_{shf}(x)$ is monomial (having no zero entries $x_i\in\mathbb{F}\backslash\{0\}$), the vectorization $\mathrm{vec}_{\mathbb{F}}^{\times}$ (98) becomes a polyadic isomorphism.*

Proposition 10. *If $x_i\in\mathbb{F}\backslash\{0\}$, the set of vectors with multiplication (100) becomes a nonderived $(m+1)$-ary group with the quervector $\tilde{\mathbf{v}}$ (5) (the polyadic analog the "inverse") having the form*

$$\widetilde{\mathbf{v}}(x)=\left(\frac{1}{x_2x_3\ldots x_m}\right)\mathbf{e}_1+\left(\frac{1}{x_3x_4\ldots x_mx_1}\right)\mathbf{e}_2+\ldots+\left(\frac{1}{x_mx_1\ldots x_{m-1}}\right)\mathbf{e}_m,\ x_i\in\mathbb{F}\backslash\{0\}. \tag{107}$$

Proof. To obtain the quervector $\tilde{\mathbf{v}}$, we use the nonderived $(m+1)$-ary product (100) and the definition (5), also equations (28) in the notation (102). □

Corollary 5. *The corresponding nonderived $(m+1)$-ary algebra $\mathcal{A}^{[m+1]}(\mathbb{F})$ with $x_i\in\mathbb{F}\backslash\{0\}$ becomes a division algebra $\mathcal{A}_{\mathrm{div}}^{[m+1]}(\mathbb{F})$ which is polyadically isomorphic to the division algebra of monomial $m\times m$ matrices, and the cyclic shift weighted matrices (101), see (11) and Theorem 3.*

Let us consider a simple example.

Example 9. *In any three-dimensional vector space $\mathcal{V}_{\mathbb{R}_3}$ over $\mathbb{R}_3$ (of any nature, with no additional structures, such as an inner product, etc., needed), we can define the nonderived associative 4-ary product of vectors*

$$\boldsymbol{\mu}_{\star}^{[4]}\left[\mathbf{v}(x'),\mathbf{v}(x''),\mathbf{v}(x'''),\mathbf{v}(x'''')\right]=(x_1'x_2''x_3'''x_1'''')\mathbf{e}_1+(x_2'x_3''x_1'''x_2'''')\mathbf{e}_2+(x_3'x_1''x_2'''x_3'''')\mathbf{e}_3. \tag{108}$$

The vector space $\mathcal{V}_{\mathbb{R}_3}$ equipped with the product (108) becomes a nonderived 4-ary algebra $\mathcal{A}^{[4]}(\mathbb{R})$ over $\mathbb{R}$. Its nonzero structure constants (103) are

$$f_{1231}^{1}=1,\quad f_{2312}^{2}=1,\quad f_{3123}^{3}=1. \tag{109}$$

In the general case, $\mathcal{A}^{[4]}(\mathbb{R})$ contains zero divisors, but for all nonzero coordinates $x_i\in\mathbb{R}\backslash\{0\}$, the algebra becomes the division algebra $\mathcal{A}_{\mathrm{div}}^{[4]}(\mathbb{R})$. The polyadic unit is $\mathbf{v}(1)=\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$, and each element $\mathbf{v}(x)$ has a quervector $\widetilde{\mathbf{v}}(x)$ (polyadic analog of the binary inverse) determined by the equation (followed from (5))

$$\boldsymbol{\mu}_{\star}^{[4]}\left[\mathbf{v}(x),\mathbf{v}(x),\mathbf{v}(x),\widetilde{\mathbf{v}}(x)\right]=\mathbf{v}(x), \tag{110}$$

such that

$$\widetilde{\mathbf{v}}(x)=\frac{1}{x_2x_3}\mathbf{e}_1+\frac{1}{x_3x_1}\mathbf{e}_2+\frac{1}{x_1x_2}\mathbf{e}_3,\quad x_i\in\mathbb{R}\backslash\{0\}. \tag{111}$$

As a simple numerical example, take four vectors with all nonvanishing coordinates, then each one has a quervector with respect to the 4-ary product of vectors $\widetilde{\mathbf{v}}$ *(110)*

$$\mathbf{v}' = \mathbf{v}(x') = 2\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_3, \quad \widetilde{\mathbf{v}}' = \frac{1}{12}\mathbf{e}_1 + \frac{1}{8}\mathbf{e}_2 + \frac{1}{6}\mathbf{e}_3, \tag{112}$$

$$\mathbf{v}'' = \mathbf{v}(x'') = 3\mathbf{e}_1 + 1\mathbf{e}_2 + 5\mathbf{e}_3, \quad \widetilde{\mathbf{v}}'' = \frac{1}{5}\mathbf{e}_1 + \frac{1}{15}\mathbf{e}_2 + \frac{1}{3}\mathbf{e}_3, \tag{113}$$

$$\mathbf{v}''' = \mathbf{v}(x''') = 4\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3, \quad \widetilde{\mathbf{v}}''' = \frac{1}{15}\mathbf{e}_1 + \frac{1}{20}\mathbf{e}_2 + \frac{1}{12}\mathbf{e}_3, \tag{114}$$

$$\mathbf{v}'''' = \mathbf{v}(x'''') = 5\mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3, \quad \widetilde{\mathbf{v}}'''' = \frac{1}{6}\mathbf{e}_1 + \frac{1}{10}\mathbf{e}_2 + \frac{1}{15}\mathbf{e}_3. \tag{115}$$

The 4-ary product (108) of the vectors (112)–(115) is

$$\boldsymbol{\mu}_\star^{[4]}\left[\mathbf{v}', \mathbf{v}'', \mathbf{v}''', \mathbf{v}''''\right] = 50\mathbf{e}_1 + 180\mathbf{e}_2 + 75\mathbf{e}_3. \tag{116}$$

Thus, we have shown that, in any m-dimensional vector space $\mathcal{V}_{\mathbb{F}}$ over a field $\mathbb{F}$, we can introduce a new nonderived polyadic (or $(m+1)$-ary) product of vectors $\boldsymbol{\mu}_\star^{[m+1]}$ (100). The corresponding $(m+1)$-ary algebra is associative and becomes a division algebra for all vectors with nonvanishing coordinates, and it is isomorphic to the polyadic division algebras obtained by the matrix polyadization procedure (Section 4).

8. Polyadic Imaginary Division Algebras

Here, we introduce a non-matrix polyadization procedure which allows us to obtain ternary division algebras from ordinary binary normed division algebras. Let us exploit the notations of the concrete hyperembedding approach from Section 6.2. First, we show that some version of ternary division algebra structure can be obtained by a new iterative process without introducing additional variables.

Definition 8. *We define the ternary "imaginary tower" of algebras by*

$$\overline{\mathbb{A}}^{[3]}_{CD,\ell+1} = \mathbb{A}_{CD}(\boldsymbol{i}_\ell)\cdot\boldsymbol{i}_{\ell+1} = \left\langle\left\{\boldsymbol{z}_{(\ell)}(\boldsymbol{i}_\ell)\cdot\boldsymbol{i}_{\ell+1}\right\} \mid (+), \overline{\mu}^{[3]}_{\ell+1}\right\rangle, \boldsymbol{i}_\ell^2 = -1, \boldsymbol{i}_{\ell+1}\boldsymbol{i}_\ell = -\boldsymbol{i}_\ell\boldsymbol{i}_{\ell+1}, \tag{117}$$

where $\ell \geq 0$.

The multiplication $\overline{\mu}^{[3]}_{\ell+1}$ in (117) is nonderived ternary, because $\boldsymbol{i}_\ell^2 \nsim \boldsymbol{i}_\ell$, but $\boldsymbol{i}_\ell^3 = -\boldsymbol{i}_\ell$ for $\ell \geq 1$.

Theorem 7. *If the initial algebra* $\mathbb{A}_{CD}(\boldsymbol{i}_\ell)$ *is a normed division algebra, then its iterated imaginary version* $\overline{\mathbb{A}}^{[3]}_{CD,\ell+1}$ *(117) is the ternary division algebra of the same (initial) dimension* $D(\ell)$ *and norm.*

Proposition 11. *The ternary algebras* $\overline{\mathbb{A}}^{[3]}_{CD,\ell+1}$ *are not subalgebras of the initial algebras* $\mathbb{A}_{CD}(\boldsymbol{i}_\ell)$.

Proof. This is because the multiplications in the above algebras have different arities, despite the underlying sets of imaginary algebras being subsets of the corresponding initial algebras. □

Let us now present the concrete expressions for the initial division algebras.

8.1. Complex Number Ternary Division Algebra

The first algebra (with $\ell = 0$) is the ternary division algebra of pure imaginary complex numbers having the dimension $D(0) = D(\mathbb{R}) = 1$

$$\overline{\mathbb{A}}^{[3]}_{CD,1} \equiv \overline{\mathbb{C}}^{[3]} = \mathbb{R}\cdot\boldsymbol{i}_1, \;\; \boldsymbol{z}_{(1)} = b\boldsymbol{i}_1 \in \mathbb{C}, \;\; \boldsymbol{i}_1^2 = -1, \;\; b \in \mathbb{R}, \tag{118}$$

which is unitless. The multiplication in $\overline{\mathbb{C}}^{[3]}$ is ternary nonderived and commutative

$$\overline{\mu}_1^{[3]}\left[z'_{(1)},z''_{(1)},z'''_{(1)}\right]=-b'b''b'''\boldsymbol{i}_1\in\overline{\mathbb{C}}^{[3]}. \tag{119}$$

The norm is the absolute value in $\mathbb{C}$ (module) $\left\|z_{(1)}\right\|_{(1)}=|b|$, and it is ternary multiplicative (from (119))

$$\left\|\overline{\mu}_1^{[3]}\left[z'_{(1)},z''_{(1)},z'''_{(1)}\right]\right\|_{(1)}=\left\|z'_{(1)}\right\|_{(1)}\left\|z''_{(1)}\right\|_{(1)}\left\|z'''_{(1)}\right\|_{(1)}\in\mathbb{R}, \tag{120}$$

such that the corresponding map $\overline{\mathbb{C}}^{[3]}\to\mathbb{R}$ is a ternary homomorphism. The querelement of $z_{(1)}$ (5) is now defined by

$$\overline{\mu}_1^{[3]}\left[z_{(1)},z_{(1)},\widetilde{z}_{(1)}\right]=z_{(1)}, \tag{121}$$

which gives

$$\widetilde{z}_{(1)}=\frac{1}{z_{(1)}}=-\frac{\boldsymbol{i}_1}{b},\quad b\in\mathbb{R}\backslash\{0\}. \tag{122}$$

Thus, $\overline{\mathbb{C}}^{[3]}$ is a ternary commutative algebra, which is indeed a division algebra, because each nonzero element has a querelement, i.e., it is invertible, so all equations of type (121) with different $z's$ have a solution.

8.2. Half-Quaternion Ternary Division Algebra

The next iteration case ($\ell=1$) of the "imaginary tower" (117) is more complicated and leads to pure imaginary ternary quaternions of dimension $D(1)=D(\mathbb{C})=2$

$$\begin{gathered}\overline{\mathbb{A}}_{CD,2}^{[3]}\equiv\overline{\mathbb{H}}^{[3]}=\mathbb{C}(\boldsymbol{i}_1)\cdot\boldsymbol{i}_2,\quad z_{(2)}=(c+d\boldsymbol{i}_1)\boldsymbol{i}_2=c\boldsymbol{i}_2+d\boldsymbol{i}_1\boldsymbol{i}_2\in\overline{\mathbb{H}}^{[3]},\\ \boldsymbol{i}_1^2=\boldsymbol{i}_2^2=-1,\quad \boldsymbol{i}_1\boldsymbol{i}_2=-\boldsymbol{i}_2\boldsymbol{i}_1,\quad \boldsymbol{i}_1\in\mathbb{C},\quad \boldsymbol{i}_2\in\mathbb{H}\setminus\mathbb{C},\quad c,d\in\mathbb{R}.\end{gathered} \tag{123}$$

We can informally call $\overline{\mathbb{H}}^{[3]}$ the ternary algebra of imaginary "half-quaternions", because in the standard notation (see Example 7) $z_{(2)}$ from (123) is $\boldsymbol{q}=a+b\boldsymbol{i}+c\boldsymbol{j}+d\boldsymbol{k}\overset{a=0,b=0}{\longrightarrow}\boldsymbol{q}_{half}=c\boldsymbol{j}+d\boldsymbol{k}$. The nonderived ternary algebra $\overline{\mathbb{H}}^{[3]}$ is obviously unitless. The multiplication of the half-quaternions $\overline{\mathbb{H}}^{[3]}$

$$\begin{aligned}\overline{\mu}_2^{[3]}\left[z'_{(2)},z''_{(2)},z'''_{(2)}\right]=z'_{(2)}\cdot z''_{(2)}\cdot z'''_{(2)}=&\left(d'c''d'''-c'd''d'''-d'd''c'''-c'c'c'''\right)\boldsymbol{i}_2\\&+\left(c'd''c'''-d'c''c'''-c'c''d'''-d'd''d'''\right)\boldsymbol{i}_1\boldsymbol{i}_2\in\overline{\mathbb{H}}^{[3]},\end{aligned} \tag{124}$$

is ternary nonderived (i.e., the algebra is closed with respect to the product of three elements, but not fewer), noncommutative, and totally ternary associative (4)

$$\begin{aligned}\overline{\mu}_2^{[3]}\left[\overline{\mu}_2^{[3]}\left[z'_{(2)},z''_{(2)},z'''_{(2)}\right]z''''_{(2)},z'''''_{(2)}\right]&=\overline{\mu}_2^{[3]}\left[z'_{(2)},\overline{\mu}_2^{[3]}\left[z''_{(2)},z'''_{(2)}z''''_{(2)}\right],z'''''_{(2)}\right]\\&=\overline{\mu}_2^{[3]}\left[z'_{(2)},z''_{(2)},\overline{\mu}_2^{[3]}\left[z'''_{(2)}z''''_{(2)},z'''''_{(2)}\right]\right].\end{aligned} \tag{125}$$

The norm is defined by the absolute value in the quaternion algebra $\mathbb{H}$ in the standard way

$$\left\|z_{(2)}\right\|_{(2)}=\left|z_{(2)}\right|=\sqrt{c^2+d^2}. \tag{126}$$

The norm (126) is ternary multiplicative

$$\left\|\overline{\mu}_2^{[3]}\left[z'_{(2)}, z''_{(2)}, z'''_{(2)}\right]\right\|_{(2)} = \left\|z'_{(2)}\right\|_{(2)}\left\|z''_{(2)}\right\|_{(2)}\left\|z'''_{(2)}\right\|_{(2)} \in \mathbb{R}, \tag{127}$$

such that the corresponding map $\overline{\mathbb{H}}^{[3]} \to \mathbb{R}$ is a ternary homomorphism.

Proposition 12. *The ternary analog of the sum of two squares' identity is*

$$\begin{aligned}&(d'c''d''' - c'd''d''' - c'c'c''' - d'd''c''')^2 + (c'd''c''' - d'c''c''' - c'c''d''' - d'd''d''')^2 \\ &= \left((c')^2 + (d')^2\right)\left((c'')^2 + (d'')^2\right)\left((c''')^2 + (d''')^2\right).\end{aligned} \tag{128}$$

Proof. It follows from the ternary multiplication formula (124) and the multiplicativity (127) of the norm (126). □

Remark 8. *The ternary sum of two squares' identity (128) cannot be derived from the binary sum of two squares' identity (Diophantus, Fibonacci) or from Euler's sum of four squares' identity, while it can be considered an intermediate (triple) identity.*

The querelement of $z_{(2)}$ (5) is now defined by

$$\overline{\mu}_2^{[3]}\left[z_{(2)}, z_{(2)}, \widetilde{z}_{(2)}\right] = z_{(2)}, \tag{129}$$

which gives

$$\widetilde{z}_{(2)} = -\frac{c\boldsymbol{i}_2 + d\boldsymbol{i}_1\boldsymbol{i}_2}{c^2 + d^2} \in \overline{\mathbb{H}}^{[3]}, \quad c^2 + d^2 \neq 0, \quad c, d \in \mathbb{R}. \tag{130}$$

It follows from (130) that $\overline{\mathbb{H}}^{[3]}$ (123) is a noncommutative nonderived ternary algebra, which is indeed a division algebra, because each nonzero element $z_{(2)}$ has a querelement $\widetilde{z}_{(2)}$, i.e., it is invertible, and equations of type (121) have solutions.

8.3. Half-Octonion Ternary Division Algebra

The next case ($\ell = 2$) gives pure imaginary ternary octonions of dimension $D(2) = D(\mathbb{H}) = 4$

$$\begin{aligned}&\overline{\mathbb{A}}_{CD,3}^{[3]} \equiv \overline{\mathbb{O}}^{[3]} = \mathbb{H}(\boldsymbol{i}_1, \boldsymbol{i}_2) \cdot \boldsymbol{i}_3, z_{(3)} = (a + b\boldsymbol{i}_1 + c\boldsymbol{i}_2 + d\boldsymbol{i}_1\boldsymbol{i}_2)\boldsymbol{i}_3 = a\boldsymbol{i}_3 + b\boldsymbol{i}_1\boldsymbol{i}_3 + c\boldsymbol{i}_2\boldsymbol{i}_3 + d\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3, \\ &\quad \boldsymbol{i}_1^2 = \boldsymbol{i}_2^2 = \boldsymbol{i}_3^2 = -1, \quad \boldsymbol{i}_1\boldsymbol{i}_2 = -\boldsymbol{i}_2\boldsymbol{i}_1, \quad \boldsymbol{i}_1\boldsymbol{i}_3 = -\boldsymbol{i}_3\boldsymbol{i}_1, \quad \boldsymbol{i}_2\boldsymbol{i}_3 = -\boldsymbol{i}_3\boldsymbol{i}_2, \\ &\quad \boldsymbol{i}_1 \in \mathbb{C}, \quad \boldsymbol{i}_2 \in \mathbb{H} \setminus \mathbb{C}, \quad \boldsymbol{i}_3 \in \mathbb{O} \setminus \mathbb{H}, \quad a, b, c, d \in \mathbb{R}.\end{aligned} \tag{131}$$

We informally can call $\overline{\mathbb{O}}^{[3]}$ the nonderived ternary algebra of imaginary "half-octonions", which is obviously unitless. In the standard notation (with seven imaginary units $\boldsymbol{e}_1 \dots \boldsymbol{e}_7$), the "half-octonion" is

$$\boldsymbol{o}_{half} = z_{(3)} = a\boldsymbol{e}_4 + b\boldsymbol{e}_5 + c\boldsymbol{e}_6 + d\boldsymbol{e}_7. \tag{132}$$

Remark 9. *It is well known (see, e.g., [56,57]), that the algebra of octonions $\mathbb{O}$ is not a field, but a special nonassociative loop (quasigroup with an identity), the Moufang loop (see, e.g., [58,59]). Because the ternary algebra $\overline{\mathbb{O}}^{[3]}$ of imaginary "half-octonions" is unitless, it cannot be a ternary loop [60].*

Recall that the binary algebra of ordinary octonions $\mathbb{O} = \langle\{z\} \mid (+), (\bullet)\rangle$ is not associative, and therefore, a triple product in $\mathbb{O}$ is not unique, because $(z' \bullet z'') \bullet z''' \neq$

$z' \bullet (z'' \bullet z''')$, $z \in \mathbb{O}$. We introduce the ternary multiplication for the "half-octonions" $z_{(3)} \in \overline{\mathbb{O}}^{[3]}$ (132) in the unique way as the following arithmetic mean

$$\overline{\mu}_3^{[3]}\left[z'_{(3)}, z''_{(3)}, z'''_{(3)}\right] = \frac{\left(z'_{(3)} \bullet z''_{(3)}\right) \bullet z'''_{(3)} + z'_{(3)} \bullet \left(z''_{(3)} \bullet z'''_{(3)}\right)}{2}. \tag{133}$$

Proposition 13. *The nonderived nonunital ternary algebra of imaginary "half-octonions" (131)*

$$\overline{\mathbb{O}}^{[3]} = \left\langle \left\{ z_{(3)} \right\} \mid (+), \overline{\mu}_3^{[3]} \right\rangle \tag{134}$$

is closed under the multiplication (133) and ternary associative.

Proof. The closeness of the product $\overline{\mu}_3^{[3]}$ is obvious, and the ternary total associativity of $\overline{\mu}_3^{[3]}$ (similar to (125)) follows from (133). □

In components, the multiplication (133) becomes

$$\begin{aligned}
&\overline{\mu}_3^{[3]}\left[z'_{(3)}, z''_{(3)}, z'''_{(3)}\right] \\
&= \left(a''\left(b'b''' + c'c''' + d'd'''\right) - a'''\left(b'b'' + c'c'' + d'd''\right) + a'\left(b''b''' + c''c''' + d''d'''\right) - a'a''a'''\right)\mathbf{i}_3 \\
&+ \left(b''\left(a'a''' + c'c''' + d'd'''\right) + b'''\left(a'a'' + c'c'' + d'd''\right) - b'\left(a''a''' + c''c''' + d''d'''\right) - b'b''b'''\right)\mathbf{i}_1\mathbf{i}_3 \\
&+ \left(c''\left(a'a''' + b'b''' + d'd'''\right) - c'''\left(a'a'' + b'b'' + d'd''\right) - c'\left(a''a''' + b''b''' + d''d'''\right) - c'c''c'''\right)\mathbf{i}_2\mathbf{i}_3 \\
&+ \left(d''\left(a'a''' + b'b''' + c'c'''\right) - d'''\left(a'a'' + b'b'' + c'c''\right) - d'\left(a''a''' + b''b''' + c''c'''\right) - d'd''d'''\right)\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3.
\end{aligned} \tag{135}$$

The querelement $\widetilde{z}_{(3)}$ (5) of $z_{(3)}$ from $\overline{\mathbb{O}}^{[3]}$ is defined by

$$\overline{\mu}_3^{[3]}\left[z_{(3)}, z_{(3)}, \widetilde{z}_{(3)}\right] = z_{(3)}, \tag{136}$$

where $\widetilde{z}_{(3)}$ can be on any place. In components, we obtain (cf. half-quaternions (130))

$$\widetilde{z}_{(3)} = -\frac{a\mathbf{i}_3 + b\mathbf{i}_1\mathbf{i}_3 + c\mathbf{i}_2\mathbf{i}_3 + d\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3}{a^2 + b^2 + c^2 + d^2} \in \overline{\mathbb{O}}^{[3]}, \quad a^2 + b^2 + c^2 + d^2 \neq 0, \quad a, b, c, d \in \mathbb{R}. \tag{137}$$

It follows from (137) that $\overline{\mathbb{O}}^{[3]}$ is a division algebra, because each nonzero element $z_{(3)}$ has a querelement $\widetilde{z}_{(3)}$, i.e., it is invertible.

Thus, we have constructed the noncommutative nonderived ternary division algebra of half-octonions $\overline{\mathbb{O}}^{[3]}$, which is nonunital and totally associative.

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